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VOLUME V

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THE TUTORIAL PHYSICS

VOLUME V

PROPERTIES OF MATTER

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PREFACE.

UNDER the title Properties of Matter the author has endeavoured to treat those branches of Physics which are not usually included in books on Light, Sound, Heat, or Electricity.

The book is intended for use in Secondary Schools, Technical Schools, and University Colleges, and includes all that is usually required for a Pass degree.

The treatment is experimental, and the use of advanced mathematics has been avoided; but the calculus has been used in places where the avoidance of it leads to tedious calculations.

It is hoped that this volume will fulfil a useful function by providing an elementary course of work on the Properties of Matter, from which the more intricate details and the more advanced theory are omitted.

The publishers are indebted to various firms for the loan of blocks for printing diagrams, as follows:—Cambridge Scientific Instrument Co. for Figs. 9, 45, 55, 64; Messrs. P. Harris & Co., Fig. 80; Messrs. Pye & Co., Figs. 10, 67; Messrs. Becker & Co., Figs. 16, 19, 65; Society for the Publication of Christian Knowledge (publishers of Prof. Boys's *Soap Bubbles*), Figs. 128, 140.

March, 1906.

NOTE TO THE SECOND EDITION.

The chief alterations in this edition are the insertion of methods for calculating certain moments of inertia, and also an elementary account of the bending of beams. In addition such misprints as have been reported are corrected

NOTE TO THE THIRD EDITION.

In order to afford a reasonable amount of practice in the various types of calculation required in this subject and a sufficient acquaintance with the magnitudes of various Physical Constants, a set of Miscellaneous Examples arranged in the order of the chapters in the book, and also a collection of Physical Tables have now been added at the end of the book.

The new matter inserted in the Second and Third Editions (viz. §§ 76a—76c, 135a—135e, and pp. 258—271) are the work of Dr. John Satterly, now Lecturer in the University of Toronto.

CONTENTS.

CHAPTER I.

UNITS; DIMENSIONS.

1. Units of Measurement.—For the exact description of any physical quantity it is necessary to express the standard or unit in which the quantity is measured, and also the number of times the quantity contains the unit. The numerical value depends on the magnitude of the unit selected, being greater for a small unit than it is for a large. Thus 1 day is 24 hours, 1440 minutes, or $\frac{1}{365}$ year. Here the day, hour, minute, year are different units, in terms of which it is usual to express time. The numerical values are inversely proportional to the magnitudes of the units selected as standard.

2. Fundamental and Derived Units.—For measuring different kinds of quantities different units must be used. These may be selected in any arbitrary way. Thus we measure volumes in quarts, cubic feet, or litres; force in pounds-weight or dynes.

It is not, however, very convenient to deal with some of these unrelated units. It is better to devise a set of units dependent on three fundamental units. The three fundamental units generally chosen are those of mass, length, and time. From these three all other units required in mechanics may be derived. Thus unit area is defined as the area of a square of which the side is of unit length; unit velocity is that of a body which moves over unit length in unit

time, and unit force is that which acting on a body of unit mass for unit time imparts to it unit velocity.

Such units are called derived units.

It is important to remember that derived units are connected with fundamental ones by arbitrary definitions adopted for convenience. Thus we might take the area of a circle of unit radius as the unit of area: but for most purposes the standard of a square of unit edge is preferable.

Units other than those of length, time, mass might be selected as fundamental. Thus the units of force, momentum, velocity could be chosen as fundamental, and those of length, time, mass derived from them. It is not easy, however, to get simple standards of momentum or velocity for the latter system, so that the length, time, mass system is used in almost all scientific work. On the English system of measurement, the standards of length, mass, time are the foot, pound, second, and these, with other units derived from them, are spoken of as the foot-pound-second system (F.P.S.). For most purposes, however, it is convenient to work on the centimetre-gramme-second (C.G.S.) system, in which the centimetre and gramme replace the foot and the pound of the F.P.S. system.

3. Dimensions.—The value of any unit depends on the values of the fundamental units from which it is derived. Thus, taking a foot as the standard of length, the units of area and volume will be the square foot and the cubic foot. If we take the yard as the standard of linear measurement, however, the units of area and volume will be the square yard and the cubic yard; one is 3^2 and the other 3^3 times as large as the corresponding units of the ordinary system. Area and volume are therefore said to be of 2 and 3 dimensions respectively in length. In general when any derived unit is dependent on the r^{th} power of a fundamental unit, it is of r dimensions in the fundamental.

The dimensions of velocity are 1 in length and -1 in time, for velocity is defined as the ratio of length to time. Acceleration is the ratio of a velocity to time: hence the dimensions of acceleration are -2 in time and 1 in length.

Momentum is defined as the product of mass and velocity, and has therefore the dimensions of 1 in mass, 1 in length, and -1 in time.

Force is defined by Newton's second law of motion as rate of change of momentum: its dimensions are therefore 1 in mass, 1 in length, and -2 in time.

The dimensions of the other units given in the following list should be worked out.

	Dimensions in Length.	Mass.	Time.
Velocity	1	0	−1
Acceleration	1	0	−2
Force	1	1	−2
Work, Energy	2	1	−2
Volume	3	0	0
Density	−3	1	0
Specific gravity	0	0	0
Area	2	0	0
Pressure	−1	1	−2
Surface tension	0	1	−2
Momentum	1	1	−1

The dimensions of some physical quantities (temperature, electricity, etc.) cannot be given in terms of M, L, T, *alone.*

The numerical factor by which a standard has to be multiplied to express any given quantity is inversely proportional to the magnitude of the standard. Thus in the example of Art. 1, 1, 24, 1440, and $\frac{1}{365}$ are inversely proportional to the duration of a day, hour, minute, and year respectively. It follows therefore from the definition of dimensions that if a fundamental unit be changed from L to, say, lL, then the measure of any quantity of a dimensions in L will have to be l^{-a} times as great on this system as it was on the old.

Thus suppose we wish to express an acceleration of 32 ft. per sec. per sec. in cm. per min. per min., the 32 must be multiplied by

$$\left(\frac{1}{12 \times 2{\cdot}54}\right)^{-1} \times (60)^{-(-2)},$$

i.e.

$$12 \times 2{\cdot}54 \times 60^2,$$

for acceleration is of 1 dimension in length and of -2 in time.

Another system of measurement is that in which the units selected as fundamental are those of length, time, and force. The units of length and time are the centimetre and second, the unit of force is the force with which the earth attracts a one gramme mass, *i.e.* the weight of one gramme at latitude 45° and at sea level.

If we adopt this system we get new dimensions for other units. The dimensions of velocity, acceleration, area, volume—if these are defined as above—will be the same as those already tabulated. Unit mass could be defined as the mass which acted on by a unit force has unit acceleration, so that the dimensions of mass would be -1 in Length, 2 in Time, 1 in Force. Momentum would have dimensions 1 in Time, 1 in Force.

4. Use of Dimensional Equations.—A knowledge of dimensions is often of use in solving problems.

Suppose for instance that we wish to obtain a formula for the time of swing of a simple pendulum. This can depend only on

(1) the length, l;
(2) the acceleration due to gravity, g;
(3) the mass of the bob, m;
(4) the angle of the arc through which the bob swings.

The circular measure of an angle has no dimensions.

Hence the only quantities having dimensions on which t depends are l, g, m, so that we may assume that $t = kl^a g^b m^c$, where k is a *numeric*, *i.e.* a numerical quantity (not necessarily constant) which has no dimensions.

The dimensions of the right-hand side must be the same as those of the left-hand side, *i.e.* 1 in time, 0 in length, and 0 in mass.

l is of 1 dimension in length,
g is of 1 dimension in length and -2 in time,
m is of 1 dimension in mass,
$\therefore\ kl^a g^b m^c$ is of $(a+b)$ dimensions in length,
$-2b$ dimensions in time,
c dimensions in mass,

$$\therefore \begin{cases} a+b=0, \\ -2b=1, \\ c=0 \end{cases} \quad \text{whence} \quad \begin{cases} a=\frac{1}{2}, \\ b=-\frac{1}{2}, \\ c=0. \end{cases}$$

Hence the relation must be

$$t = kl^{\frac{1}{2}}g^{-\frac{1}{2}},$$

or

$$t = k\sqrt{\frac{l}{g}}.$$

The numerical coefficient k can be found experimentally. Its value is 2π (a constant) when the angle of swing is small.

This method is one that is often useful but is not always available. Suppose for instance we wished to find a relation between the space s which a body moving with an initial velocity u and acceleration f describes in time t.

Put $$s = ku^a f^b t^c,$$

u is 1 dimension in length and -1 in time,
f is 1 dimension in length and -2 in time,
t is 1 dimension in time,

so that $ku^a f^b t^c$ is of $a + b$ dimensions in length and $-a - 2b + c$ in time,

$$\therefore a + b = 1,$$
$$a + 2b - c = 0.$$

These two equations are not sufficient to determine a, b, c, which might have several different sets of values.

Two sets are

$$a = 1, \quad b = 0, \quad c = 1,$$
$$a = 0, \quad b = 1, \quad c = 2.$$

These correspond to the formula $s = ut + \frac{1}{2}ft^2$, but it is evident that the formula could not be obtained in this way.

However, even in cases where a knowledge of dimensions is not serviceable for constructive purposes, it is often useful as a test of the accuracy of an expression or formula: for all terms involved must have exactly the same dimensions.

5. Choice of Standard Units. Unit of Length.—In choosing a standard unit—especially a fundamental unit—it is of great importance to have one which is unalterable and one with which other standards supposed to be exactly like it can readily be compared. The French Republic (1801) decreed that the unit of length should be one ten millionth of an arc of the earth measured from the equator to the pole. This length was termed the metre. It had been specially determined by a committee appointed by the French Academy of Sciences. Later and more accurate determinations of the arc were not to affect the metre,

which once determined was to be defined as the length of a certain bar of metal. The metre is now defined as the length at 0° C. of a certain platinum rod kept in the Archives at Paris. This is known to be not exactly one ten millionth of a quadrant: it is too short by about ·2 mm. A suggested unit of measurement is that of a wave length of a particular kind of light in vacuo under normal conditions. This we believe to be absolutely unalterable.

On the F.P.S. system the standard length from which the foot is derived is the yard. The yard is defined as the distance between two marks on a certain bronze bar kept by the Board of Trade when the temperature of the rod is 62° F. Several accurate copies of this are in existence.

It is not based, as the metre was intended to be, on any fixed natural quantity.

6. Unit of Mass.—The unit of length in the C.G.S. system having been selected as explained above, that of mass was associated with it, the gramme being defined as the mass of a quantity of water the volume of which is one cubic centimetre at a temperature of 4° C. The gramme is now practically defined as the mass of the one thousandth part of a certain lump of platinum-iridium kept in the Archives at Paris. This was intended to be equal to the weight of a cubic decimetre of water. The English standard of mass is the pound. This is the mass of a lump of platinum in the possession of the Board of Trade.

7. Unit of Time.—The unit of time in both the C.G.S. and the F.P.S. systems is the mean solar second, a period which is related to the period of revolution of the earth on its axis. The earth as it turns on its axis also revolves round the sun, so that the period of revolution of the earth on its axis would seem to an observer on a star to be rather less than it would be to an observer on the sun. The earth makes approximately 366¼ revolutions on its axis in the course of a year. Viewed from the sun, however, it would appear to revolve only 365¼ times (*v.* Chap. VIII.).

To illustrate this, consider the case of a cogged wheel (A) rolling round the outside of another fixed cogged wheel (B) of the same

diameter. Seen from the centre of the wheel B, A will appear to rotate once every time it rolls completely round B: seen from a point outside the system it will appear to rotate twice on the same journey. Similarly if one wheel roll round on the *inside* of another of three times its diameter, it will actually revolve only twice every time it completely travels round the larger. Viewed from the centre of the larger it appears to rotate three times.

The true period of revolution of the earth on its axis—*i.e.* the interval which elapses between two consecutive passages of the same star across the meridian—is called a sidereal day.

The interval that elapses between two consecutive passages of the sun across the meridian—*i.e.* the time between true noon one day and true noon the next—is called a solar day. The length of this is not constant, but depends on the time of the year. Its average length is called a mean solar day, and from it the second is derived. A second is therefore defined as $\frac{1}{24} \times \frac{1}{60} \times \frac{1}{60}$ of a mean solar day.

The period of revolution of the earth is probably gradually increasing as the result of a slowing down by tidal friction. The change in the length of the day is, however, not sufficient to cause any appreciable variation in the unit of time.

8. Absolute and Gravitational Units.—The unit of force is derived directly from these three fundamental units. It is defined as the force which acting on unit mass produces unit acceleration. In the C.G.S. system this force is termed a *Dyne*, in the F.P.S. system it is termed a *Poundal.* Both these units are *absolute* units and invariable. Engineers, however, often find it more convenient to take the weight of a one pound mass as the unit of force. Such a unit is termed a *gravitational* unit. Its value changes from place to place, being greater at the poles than at the equator.

Derived from the absolute units of force we have other absolute units such as those of work (joule, foot poundal) and power (watt); while from the gravitational unit of force we have gravitational units such as the foot-pound, horse-power. Absolute units are generally preferred in physical measurements.

9. Scalar and Vector Quantities.—Many quantities which are dealt with in physics and dynamics may be placed in one or other of two classes—scalars and vectors. A scalar quantity is one with which no idea of direction is associated and is completely defined by its magnitude: thus mass, volume, energy are all scalar quantities: none of them can have direction or orientation. On the other hand the effects or values of vector quantities are dependent on their directions. They can be represented by straight lines. Amongst such quantities are velocity, force, momentum, acceleration. A knowledge of the amount of a scalar quantity is all that is usually required in dynamics, but for the complete description of a vector quantity, its direction as well as its magnitude must be given. Scalar quantities of the same kind may be added together directly. The combined effect of two vector quantities cannot generally be found by direct addition in this way, but by means of the *parallelogram of vectors*, which may be enunciated as follows: If two vectors are represented by two adjacent sides of a parallelogram, the single vector to which they are equivalent can be represented by the diagonal of the parallelogram drawn through the point at which the two sides meet. The proposition known as the parallelogram of forces is a particular case of this more general theorem.

10. Multiplication of Physical Quantities.—We have often spoken above of multiplying or dividing one physical quantity by another. The use of these terms perhaps requires some explanation. In arithmetic we are accustomed to multiply two numbers together: $3 \times 7 = 21$; we also multiply quantities by numbers: 3 tons $\times$ 7 = 21 tons; but we less seldom speak of multiplying one quantity by another quantity. To speak of multiplying "*length by width*" is perfectly correct: the product is an area. The words "multiply," "product," "quotient," "ratio" have, however, wider meanings than those attached to them in pure mathematics. When one physical quantity is multiplied by another the product is a quantity different in nature from the factors. The product *force* $\times$ *distance* is termed *work* or *moment*. Sometimes the product is a quantity less familiar to us. We do not know the nature of the product *length* $\times$ *volume*: but this does not prevent us from working on such quantities.

EXAMPLES I.

1. The kinetic energy of a mass m moving with the velocity v is expressed by the product kmv^2, where k is a constant. Show that this expression is of proper dimensions, and that no other form involving m and v only is possible.

2. The velocity (v) of sound in a gas is a function of its pressure (p) and density (d). Prove $v \propto \sqrt{\frac{p}{d}}$.

3. The frequency (n) of vibration of a stretched string is a function of the tension (T), the length (l), and the mass per unit length (ρ). Prove
$$n \propto \frac{1}{l}\sqrt{\frac{T}{\rho}}.$$

4. The time of oscillation (t) of a small drop of liquid under surface tension depends only on the density (ρ), the radius (a), and the surface tension (T). Show that the period of oscillation is $k\rho^{\frac{1}{2}}a^{\frac{3}{2}}T^{-\frac{1}{2}}$, where k is a numeric.

(The full relation is $$t = 2\pi\sqrt{\frac{\rho a^3}{8T}}.$$ *v.* Art. 263.)

5. Assuming the relations between the centimetre and the foot, and between the pound and the gramme, use your knowledge of dimensions to convert $4 \cdot 2 \times 10^7$ ergs to foot-poundals.

6. Express a pressure of 1 kilogramme weight per square centimetre in pounds weight per square inch.

7. What are the dimensions of power, surface density, specific gravity, angular velocity?

8. What is meant in physics by the term—a directed quantity? Point out which of the following are directed quantities and which are not:—Volume, pressure, density, acceleration, magnetic field, mass, weight, energy.

9. If the acceleration of gravity be represented by unity, and one second be the unit of time, what must be the unit of length?

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER II.

LENGTHS AND AREAS.

11. Measurement of Length.—For measuring lengths the instrument in general use is a metre rule divided into centimetres and millimetres. By means of this, lengths such as are usually measured in a laboratory may be obtained accurately to a millimetre, while tenths of millimetres can sometimes be estimated. An accuracy such as this is often sufficient: thus if the length of a knitting needle is required it might be measured as 24·2 cm. If this is correct to the nearest millimetre the true length must lie between 24·15 cm. and 24·25 cm. and the possible error in the measurement is ± ·05 cm., *i.e.* about ·2 per cent. $\left(\frac{\cdot 05}{24 \cdot 25} \times 100\right)$. When smaller lengths are concerned accuracy to a millimetre may be insufficient. Thus an error of ·1 mm. in measuring a length of 4 mm. is an error of more than 2 per cent. This is inadmissible in exact work, so that other means of measurement are required. One of the simplest of these is the diagonal scale.

√**12. The Diagonal Scale** is represented in Fig. 1. It is divided parallel to its length into ten equal spaces and the right-hand division is also subdivided by ten equidistant diagonal lines. Suppose the distance between two points, *A* and *B*, is required. Take a pair of dividers and open them to such an angle that it can just span the length *AB*. Now transfer to the scale. Suppose the length *AB* lies between 1 and 2 inches. Place one point of the dividers at the point *E*. Suppose the other point falls on *F*. Now run the dividers across the scale so that the points begin

to trace out the parallel lines EG, FH. The line FH will cut one and only one of the diagonal lines. In the figure it cuts the sixth at the point where that is also cut by the horizontal line seventh from the top. It will be readily seen that the distance between the lines EG, FH, *i.e.* the length required, is 1·57 in.

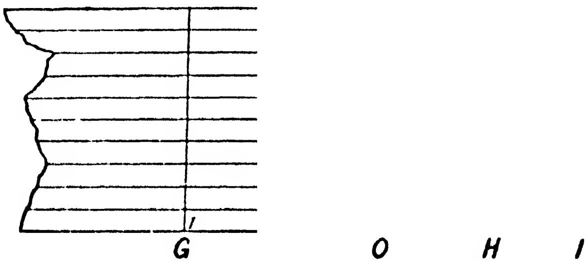


Fig. 1.

The diagonal scale is useful for measuring lengths on charts and scale drawings, but for most other purposes its adaptation is cumbrous and inaccurate.

13. Vernier.—A neater and more serviceable device is the vernier. Such an instrument is represented in Fig. 2. In this the upper part is a centimetre scale. The lower part is set off into ten equal divisions, each of which is

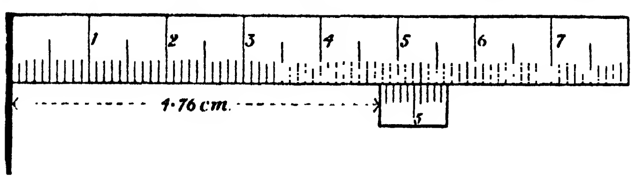


Fig. 2.

·9 mm. long. The vernier divisions are therefore less than the scale divisions by ·1 mm. Now suppose that one of the vernier divisions (say the sixth) exactly coincides with one of the millimetre divisions on the scale (say the 5·3 cm.), then the 5 on the vernier will be ·1 mm. beyond 5·2 cm., the 4 on the vernier will be ·2 mm. beyond the

5·1 cm., and so on till finally the 0 on the vernier will be ·6 mm. beyond the 4·7 cm. division.

The length indicated, *i.e.* the distance between the 0 on the scale and the 0 on the vernier, is therefore 4·76 cm. Hence to read the length on a vernier we look for the centimetres and millimetres exactly as on a metre rule, while tenths of millimetres are shown by the number of the graduation on the vernier which most nearly corresponds with a scale division.

Verniers can of course be constructed to read other fractions besides tenths, say twelfths or twentieths. Each vernier division would then be $\frac{1}{12}$ or $\frac{1}{20}$ less than a scale division, *i.e.* 12 or 20 vernier divisions would be equal to 11 or 19 scale divisions. (Fig. 3.)

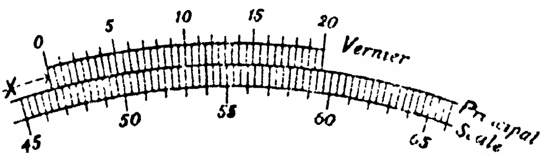


Fig. 3.

Verniers reading to minutes are often attached to the scales of theodolites, spectrometers, and other instruments used for reading angles. In such cases it is usual to divide the scale into thirds of a degree. The vernier is divided into twenty parts, so that it reads to the twentieth of a third of a degree, *i.e.* to minutes. (Fig. 3.)

The verniers so far described are forward reading verniers, and are the kind in general use. Backward reading verniers (Fig. 4) are sometimes placed on barometers. When constructed to read tenths, 10 vernier

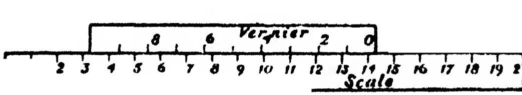


Fig. 4.

divisions must be equal to 11 scale divisions, so that each vernier division is greater than a scale division by a tenth. In figure 4 the reading is 14·3.

14. Micrometer Screw Gauge.—The ordinary vernier callipers read to tenths of a millimetre. When greater accuracy is required in the measuring of small bodies the screw gauge may be used. (Fig. 5.) This consists of the screw *AB*, the jaws and the barrel through which the screw passes. The stud at *F* may be replaced or adjusted to take up wear.

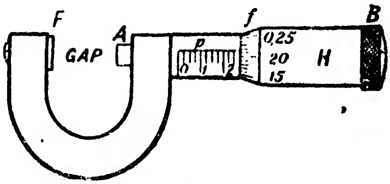


Fig. 5.

The screw *AB* must be very accurately cut, and must fill the barrel closely and without play, but at the same time should move easily. The figure represents a millimetre gauge. The pitch of the screw, *i.e.* the distance between consecutive threads, is 1 mm., so that a complete turn of the head advances the screw by that amount. Suppose the diameter of a wire is required. To use the instrument hold the wire on the stud at *F* and then advance the screw by turning the milled head *B* with thumb and finger till the wire is just nipped. The number of millimetres is then read off from the scale on the barrel, and the hundredths from the number of that graduation on the circumference of the sleeve at *f* which is opposite to the scale line on the barrel. The principle on which this and all instruments which measure by means of screws are made is that any fraction of a complete turn of the screw advances the screw by the same fraction of its pitch. Some care is required in using the screw gauge. In the first place it must be screwed home very gently on the object to be measured, which would otherwise be slightly compressed. Further, the "zero error" should be obtained, *i.e.* the reading when the screw is brought home on the stud with nothing between. This must either be added to or subtracted from all other readings obtained. An appreciable zero error nearly always exists, for stud and screw both wear slightly, and no two observers screw up with the same force. In some instruments this last source of error is almost entirely eliminated by using a ratchet

head which ceases to turn the screw when the pressure has reached a certain constant value. Readings so taken will therefore be nearly independent of the observer. If the pitch of the screw is $\frac{1}{2}$ mm. the graduated circle will be divided up into 50 parts, and the scale on the barrel must be consulted to determine whether, say, 24 indicates ·24 mm. or ·74, *i.e.* (·50 + ·24) mm.

15. The Spherometer is used to determine the radius of curvature of a surface, *e.g.* that of a mirror. The three pointed feet *A*, *B*, *C* (Fig. 6) are situated at the vertices of an equilateral triangle, and the axis of the screw *SE* is perpendicular to the plane of this triangle and passes through its centre. The height of the point *E* above the plane of *ABC* is read off by means of the pitch scale and the graduated circle (head scale), the units being marked on the scale and the hundredths on the circle—exactly as in the screw gauge.

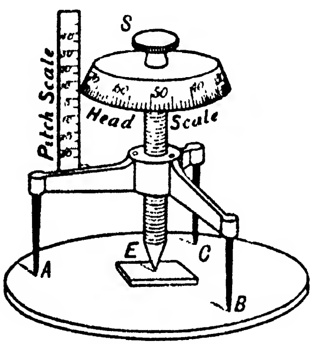


Fig. 6.

To use the spherometer raise up the screw and place the three feet on the surface of the mirror the curvature of which is required. Now gently and gradually turn down the screw till the point *E* just touches the surface. This is generally indicated by the slipping of the legs round the centre *E*. The reading, giving the height of *E* above the plane of the feet, is then taken. The radius (R) of the surface is given by the relation

$$R = \frac{l^2}{6h} + \frac{h}{2},$$

where

h = height of the point *E*,

l = length of side of the triangle formed by the three feet.

This formula may be obtained as follows:

Consider the triangle formed by the feet ABC (Fig. 7). Suppose D is the centre of this triangle, F the middle point of the side BC,

$$\text{then } AB = l,\ FC = \frac{l}{2},$$

$$\text{and } AF = \frac{\sqrt{3}}{2}l,$$

$$\text{but } AD = \frac{2}{3}AF,$$

$$\text{thus } AD = \frac{l}{\sqrt{3}}.$$

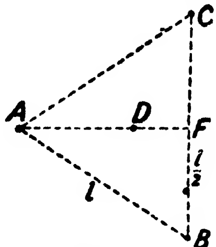


Fig. 7.

Now Fig. 8 represents a central section of the sphere of which the surface of the mirror forms a part. This section is taken through the points where the feet A, E meet the mirror. The axis of the screw will pass through the point D and the centre of the sphere, and will cut the surface again in a point L.

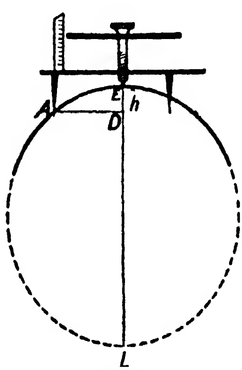


Fig. 8.

Then by the geometry of the circle

$$AD^2 = ED \cdot DL,$$

$$\therefore \left(\frac{l}{\sqrt{3}}\right)^2 = h(2R - h),$$

$$\therefore 2R = h + \frac{l^2}{3h},$$

which is the relation given above.

For most purposes the length l may be obtained with sufficient exactitude by pricking the points A, B, C on a piece of paper and measuring the distance between the pricks by dividers and scale.

A piece of flat glass is generally to be found in the case of the spherometer. This must be used to determine the zero error of the instrument.

In the above description we have supposed that the surface of the mirror was convex. If it is concave the

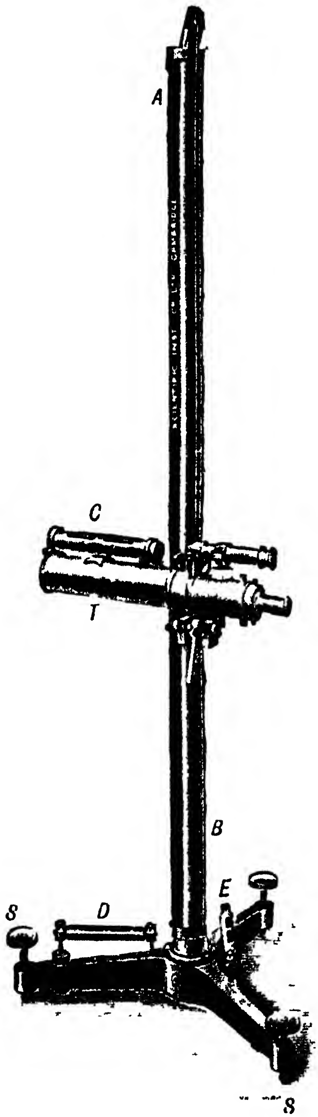


Fig. 9.

formula still holds, h in this case standing for the depth of the point O below the plane of the feet. In measuring this depth it is necessary to remember that the reading on the graduated circle must be subtracted from 100, so that if the instrument measures to hundredths of a millimetre and the circle is divided up into a hundred parts, then, when the 38 graduation stands opposite the scale at a level between the 3 and 4 marks on that scale, the reading indicated is 3·62 mm.

Some instruments have however a double ring of graduations on the circle, one for reading convex surfaces, the other for reading concave.

The instrument may be used instead of a screw gauge for finding the thickness of a small piece of glass or of a coin.

16. The Cathetometer is an instrument used to measure vertical heights. It consists of a graduated vertical rod AB carrying a slide to which is fastened a telescope T with the axis horizontal.

The base to which the vertical rod is fixed is three-legged, and carries levelling screws S, S, S and spirit levels D, E. The telescope also carries a spirit level C.)

The slide which moves up and down the rod carries a vernier, so that the distance the slide is moved up or down can be read off accurately on the scale. The telescope can be rotated round the rod AB, always remaining horizontal.

To read the difference in level between two points, P and Q say, the telescope is focussed on the first, P, and is raised or lowered till the image of P is exactly on the cross wires in the eye-piece. The vernier reading is then taken and the telescope is then arranged to bring the image of Q on the cross wires. The difference between the readings given by the vernier is equal to the difference in level between P and Q.

In using the instrument great care must be taken to ensure

(1) That the rod AB is truly vertical.

(2) That the axis of the telescope is horizontal.

(3) That the object is accurately focussed on the cross wires.

The means of securing these essentials are given in *Practical Physics* (Bower and Satterly).

17. The Travelling Microscope may be used to measure small horizontal lengths. Fig. 10 shows a convenient mounting. The microscope M is carried by an arm A which can be firmly clamped by the screw T on the rod C. S is a screw the end of which can be brought up against the end of the rod C. The distance between the threads of the screw is generally about ½ millimetre, and the head is divided into 100 divisions, so that the instrument will read to nearly ·005 mm.

Suppose it is required to measure the distance between two parallel scratches on a piece of metal. The metal is placed under the microscope with one of the scratches parallel to the cross wire of the eye-piece. The rod C is kept pressed up against the screw while the arm A is moved till the scratch and cross wires are very close together, but with the cross wire slightly to the right of the scratch.

The arm is then clamped and the screw turned till exact coincidence is obtained. The reading of the head is then taken. Then the microscope is focussed on the other

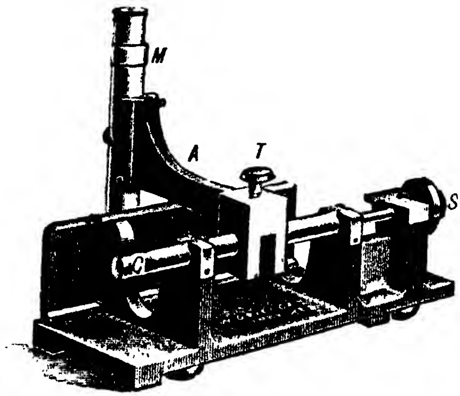


Fig. 10.

mark in exactly the same way and a second reading obtained. The difference between the two gives the distance between the marks.

A vernier is used in place of the screw in a *Vernier Microscope.*

18. The Areas of Plane Surfaces.—The measurement of the areas of many plane figures often requires nothing more than a knowledge of their linear dimensions. We shall assume that the following geometrical results are known:

Area of a rectangle	= length × breadth.
,, ,, parallelogram	= length × altitude.
,, ,, triangle	= $\frac{1}{2}$ base × altitude.
,, ,, trapezium	= $\frac{1}{2}$ sum of parallel sides × altitude.
,, ,, circle	= $\pi \times (\text{radius})^2$.
,, ,, ellipse	= π × rectangle of semi-axes.
,, ,, curved surface of a cone	= $\frac{1}{2}$ length of slope side × circumference of base.
,, ,, surface of a sphere	= $4\pi\,(\text{radius})^2$.

19. Simpson's Rules.—The areas of many plane figures may be obtained by dividing them up into triangles or squares. Thus a quadrilateral figure may conveniently be divided into two triangles, and curvilinear plane figures may be traced on millimetre paper and the number of enclosed squares counted.

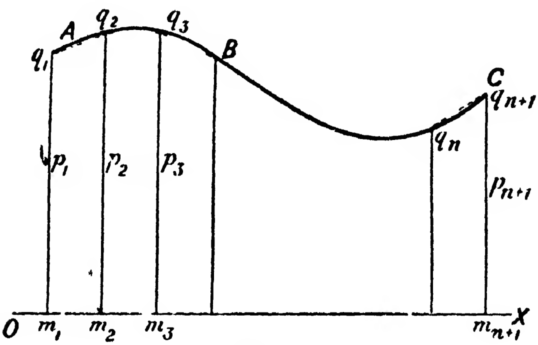


Fig. 11.

The following method is often convenient. Suppose it is required to find the area between the curve ABC and the line OX. Divide the area up into n parts by the $n+1$ evenly spaced ordinates $p_1, p_2, p_3, \ldots$ which meet the curve in $q_1, q_2, q_3, \ldots$ and the line OX in $m_1, m_2, m_3 \ldots .$ Then $m_1m_2 = m_2m_3 = \ldots = b$, say.

Join $q_1q_2, q_2q_3, \ldots q_nq_{n+1}$.

The area of the trapezium $q_1m_1m_2q_2 \quad = \frac{1}{2}(p_1 + p_2)b$,

" " " $q_2m_2m_3q_3 \quad = \frac{1}{2}(p_2 + p_3)b$,

" " " $q_3m_3m_4q_4 \quad = \frac{1}{2}(p_3 + p_4)b$,

" " " $q_nm_nm_{n+1}q_{n+1} = \frac{1}{2}(p_n + p_{n+1})b$.

Hence by addition the area of the rectilineal figure

$$m_1q_1q_2q_3 \ldots q_{n+1}m_{n+1} = \tfrac{1}{2}b(p_1 + 2p_2 + 2p_3 + \ldots 2p_n + p_{n+1})$$
$$= b\left(p_2 + p_3 \ldots p_n + \frac{p_1 + p_{n+1}}{2}\right).$$

Now if the ordinates are fairly close together, the area of the curvilinear figure differs little from that of the rectilinear. Hence we have the following rule:—

Divide up the area by equally spaced ordinates, measure the ordinates; add the mean of the first and last ordinates to the sum of those between; multiply the total so formed by the distance between consecutive ordinates.

The above rule is sufficiently accurate for most purposes. A nearer approximation is obtained by adding to the sum of the first and last ordinates twice the third, fifth, seventh . . . ordinates, and four times the second, fourth, sixth . . ., and then multiplying by one-third of the distance between consecutive ordinates. For this rule the number of ordinates $(n+1)$ must be *odd*.

Another method is to cut out the figure in assay lead foil or paper of uniform thickness. Weigh this. Weigh also a rectangle cut from the same sheet and calculate the area. The ratio of the weights equals the ratio of the areas.

20. **The Planimeter.**—This is an instrument that records directly the area of any plane figure.

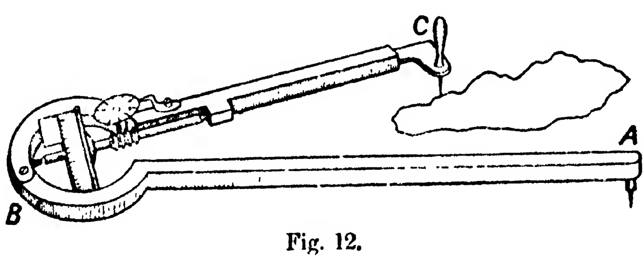


Fig. 12.

The two arms AB, BC are freely jointed at B. In the elbow at B is placed a graduated wheel which revolves round an axis parallel to the length of the arm BC.

In use the instrument rests on the paper on which the curve is traced, touching it with the spikes A and C and with the lowest part of the circumference of the wheel. The point A remains fixed, while C is made to travel round the boundary of the figure in such a way that the two arms finally return to their initial positions. Motion at right

angles to BC is recorded on the wheel, but motion along BC does not affect it. The difference between the readings of the wheel before and after the spike C has travelled round the figure gives the area.

The theory of the instrument is explained in Williamson's Integral Calculus.

Examples II.

1. Find a formula for the radius of a spherical surface in terms of the height of the middle leg above the other three, and the distance from the middle leg to each of the others.

2. In finding the radius of a spherical surface by the method of Art. 15, an error of 1% is made in measuring (1) h, (2) l. By how much will the calculated value of R be wrong in each case when h is about 1 mm. and l about 4 cm.?

3. Discuss the use of the diagonal scale. What accuracy is obtainable by it?

4. A backward reading vernier measures to twentieths of a centimetre. What is the length of each graduation on it?

5. What is a vernier? How would you construct a vernier to enable you to make barometric readings to 0·002 of an inch, the graduations of the scale being twentieths of an inch?

6. How may the accuracy of a spirit level be tested? A spirit level is in the form of the arc of a circle of 3 ft. radius. If the level be placed on an incline of 1 in 25 by how much will the bubble be displaced from the centre?

Practical Exercises.

1. Determine the radius of a spherical mirror by the spherometer and verify by an optical method.

2. Measure the diameter of a thin copper wire by the screw gauge. Weigh a known length and so find the density of copper.

3. Use a vernier microscope to test the accuracy of the graduations on a rule.

4. Describe a circle of 3″ radius on squared millimetre paper, and by counting squares determine its area. Find the ratio of this area to that of the square on its radius.

5. Describe a sine curve and find the area enclosed between the initial line and one undulation.

6. Measure the diameter of the bore of a piece of thermometer tubing.

7. Measure 10 inches on a given inch rule by a centimetre scale, and so compare the centimetre and inch.

(For further examples on this chapter see Miscellaneous Examples, p. 258.)

CHAPTER III.

MATTER; MASS.

21. Matter.—"All we know about matter relates to the series of phenomena in which energy is transferred from one portion of matter to another, till in some part of the series our bodies are affected, and we become conscious of a sensation."[1]

Hence, though we know many of the properties of matter, we know very little of its nature. Two pieces of matter attract one another: the law of attraction is known, but the mechanism by which this attraction is exerted is unknown. Matter has inertia, but we cannot say why it has this property. Electricity also has a kind of inertia. Perhaps inertia in matter and electrical inertia may be identical. If so, we can explain one of the properties of matter in the language of electricity. It is somewhat remarkable that though electricity has scarcely been thought of till recent years, its nature is in some respects much better known to us than that of matter, and it seems possible that in a few years we may look on electricity rather than on matter as the basis of the universe.

Some of the properties common to matter in all states are these:

(1) Matter is *indestructible*, uncreatable.

(2) *Inertia*. Matter in motion has energy. Force must be applied to matter to produce motion. Newton's First Law states that every body keeps its state of rest or uniform motion in a straight line except in so far as it is compelled to change that state by forces impressed on it.

[1] J. C. Maxwell, *Matter and Motion*.

(3) *Mutual attraction.* All particles of matter attract one another.

(4) *Divisibility.* Matter can be split up into portions so small as to be quite invisible. The divisibility, however, appears to be limited.

(5) Matter is *porous*, that is to say that matter in one form can pass into a space apparently filled by matter in another. Thus water can pass through gold or lead, mercury through most metals, gases through hot iron.

The volume of a mixture is often less than the sum of the volumes of its components. Thus if alcohol and water be poured separately into a measuring cylinder, the level of the mixture falls when the two are stirred together.

(6) *Elasticity.* The volume of any portion of matter may be diminished, but forces are called into play which resist the change. When the forces applied are withdrawn the matter regains its former volume. Solids can resist changes of shape and volume impressed upon them and can recover from them. This power of recovery is called Elasticity.

(7) Matter is the vehicle of *energy.* Energy cannot exist except in connection with matter.

Some of these properties require further discussion.

22. The Atomic Hypothesis.—As we have stated above, matter would appear to be divisible to an almost indefinite extent. A single drop of oil will spread over a large expanse of water, and its presence can be shown by the colours it produces in the sunlight. Scents may be detected when the quantity of the material present is beyond the range of the most sensitive balance and the most delicate chemical reagents. The presence of mercury vapour can be recognised in a Torricellian vacuum. Yet we cannot regard matter as indefinitely divisible. Our reasons for this are many. The kinetic theory of gases, explained in Chapter XI., requires us to believe that a gas is not an expansible or continuous medium, but an aggregation of particles far removed from one another. The laws of chemistry, the solution of one

substance in another, the passage of electricity through gases and liquids, all seem explicable only on the hypothesis that matter is divisible into discrete particles. The atoms of the chemist are the smallest particles into which he can split up matter. Whether these small bodies are the same as the ultimate particles demanded by a physicist is immaterial, but at present it seems probable that the smallest chemical atom is divisible under the great stresses produced in vacuum tubes into many parts, and that these corpuscles or electrons, as they are termed, are the ultimate basis of all matter.

23. Ether.—Light and heat pass from the sun to the earth. They are not propagated instantaneously, but take a definite time for the journey. Hence light and energy of other forms exist as radiant energy in interplanetary space. The vehicle must be material. This leads us to think that matter of some form must fill the gap between sun and earth. We know very little of this form of matter: perhaps it is not right to call it matter, at any rate it differs vastly from the gross matter we can perceive with any of our senses. We call it ether, and look upon it sometimes as a continuous incompressible fluid, sometimes as a highly elastic solid, the vibrations of which constitute light and radiant heat.

Other phenomena postulate the existence of an ether. It seems inconceivable that the sun could attract a planet except by means of some connecting medium. Electric and magnetic actions also require something that will support great stresses in apparently void space. Some of the properties of ether appear to be mutually exclusive. The planets would appear to revolve round the sun unimpeded by the ether, so that the ether must be more fluid than the most perfect gas. The transverse vibrations to which it is subjected in the passage of light necessitate enormous rigidity. It may help us to reconcile these views if we consider the behaviour of pitch. This behaves as a brittle solid when strains are suddenly impressed upon it; yet, given time, its own weight will make it flow as a liquid.

24. Vortex Theory of Matter.—Many hypotheses have been formulated as to the nature of matter and of ether. Some of these must be briefly described. The first is the vortex theory suggested some years ago by Lord Kelvin.

Every one knows the appearance of a vortex ring. Such a ring can be produced by suddenly expelling a puff of gas through a circular orifice. They may be produced from the bowl of a pipe or from the lips. For demonstration purposes the usual method of procedure is to cover one end of a box with a cloth, cut a round hole in the other end and lead in through one side a stream of hydrochloric acid gas, through the other a stream of ammonia. The gases mix and form a dense cloud. A smart tap on the flexible back causes a ring to be projected through the mouth. Such rings may be made to travel right across a room. Their rotary movement endows them with elasticity of shape; two rings will bounce away from one another. On the vortex theory of matter, an atom is regarded as a vortex ring of ether. Such rings could not be created, yet could not be destroyed. Smoke rings are created by friction: the middle part of the puff passes out faster than the outer boundary which is retarded by the rough edge of the box, but we regard ether as a frictionless fluid, and in such a fluid rotational motion can neither be created nor destroyed. The hypothesis then accounts quite easily for the indestructibility of matter. Other properties, however, are not so easily explained. In particular we have no evidence that the vortex rings would attract one another, so that there appears no reason for universal gravitation. The vortex theory cannot now be accepted in its crude form.

25. The Granular Hypothesis.—Amongst other theories of the universe, that of Osborne Reynolds was favourably considered at one time. It is well known that granular particles can be packed together in two distinct configurations. In one each sphere touches twelve neighbours, in the other only six; so that the total volume occupied by a given number of spheres is much greater in the latter position than in the former. In Reynolds' theory the ether is supposed to consist of excessively

small hard spheres. In free space these spheres are packed together as closely as possible. At other places they are less closely packed, and this deviation from the usual condition of the granular ether is supposed to constitute an atom. The movement of an atom is then simply a movement of a rift in the ether, the ether particles passing through the interior of the atom to the other side, leaving another similar space in their rear.

Conservation of matter is thus a conservation of form rather than of material.

The theory taxes the imagination somewhat severely. The mean pressure in the medium is 750,000 tons-wt. per square inch. The diameter of each granular particle is only the seven hundred thousand millionth part of a wave length of violet light. The mean density of the medium is ten thousand times greater than that of water.

The author however claims that his theory accounts for universal gravitation, cohesion, elasticity, the electric and magnetic properties of matter, the transmission of light, and many things besides.

A short account of this "Inversion of Ideas" is given in his *Structure of the Universe.*

26. The Electron Theory.—Some other views on the nature of matter must be briefly sketched.

We know that energy must be spent to set up an electric current: acceleration of an electric charge implies the existence of force of some kind. A charge of electricity is therefore associated with inertia. Prof. J. J. Thomson suggests that an atom is a collection of positive and negative charges of electricity: it consists of a large quantity of positive electricity with small negative "corpuscles" describing orbits in it. This hypothesis represents the mass of any atom as due to these electric charges which carry the ether along with them, so that all mass, momentum, kinetic energy belong really to the ether.

If a number of magnetised needles be arranged to float upright with their S. poles upwards in a basin of water above which is placed a strong magnet with its N. pole downwards, the magnets arrange themselves round the N. pole:

their equilibrium position being due to the forces of attraction due to the N. and the repulsion between themselves. Now some of these groupings have a remarkable similarity. For example, a group of 18 magnets may be made up of 1 at the centre, a middle ring of 6, and an outer ring of 11; a 33 group is similar, but has a fourth ring of 15 magnets. Thus we have a series of

$$1, \quad 1+6, \quad 1+6+11, \quad 1+6+11+15,$$

such that each group resembles the one before it, but has an extra ring. A similar sort of grouping would exist in negative electrons revolving in a sphere of positive electricity. The repetition of the groups is suggestive of an explanation of Mendeléef's Periodic Classification of the Chemical Elements. The theory accounts for most of the phenomena of radioactivity. (See *Higher Text-Book of Magnetism and Electricity*, Ch. XXXIV.)

Another conception of the atom is due to Lord Kelvin. He suggests as a model of the atom a series of heavy concentric shells connected together by springs: this arrangement would give an atom a set of definite vibration periods.

Lastly, we have the idea of an ether like an elastic solid such as caoutchouc: an atom being merely a discontinuity, a place where the ether has been rifted and joined up again--not truly, but with a strain space left.

27. Solids; Fluids.—Matter is divided into two classes: solids and fluids. A fluid is defined as matter in such a state that it yields continuously to any shearing stress applied to it however small that stress may be. By a shearing stress is meant a combination of forces which act in such a way that the reaction between two parts of a body separated by a plane area is not normal to this plane. Take for instance a heap of sand (Fig. 13). Consider two portions of it cut off by a plane AB inclined to the horizon. The action of one portion on the other may be resolved into two sets of forces: one in the plane AB, the other normal to AB. The former constitutes a *tangential* or *shearing stress*, tending

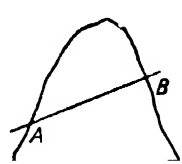


Fig. 13.

to make the part above AB to slide down over the lower part. In the case of sand—a collection of solid particles—the tendency to slide, if not too great, may be resisted indefinitely. A heap of sand could exist for ever. Sometimes a heap of fluid collapses at once. A heap of treacle or tar flows away more slowly. A block of pitch or sealing wax will take months to run perceptibly. Pitch and sealing wax, however, do flow in time: they give continuously, though very slowly to the smallest stress applied.

The fluidity of pitch is shown by the following experiment made by Kelvin. He took a block of pitch and supported it on corks placed at the corners. On the top of the block he placed a few lumps of lead. In the course of time the corks floated up to the top of the block and the lead sank down to the bottom.

If a rod of sealing wax is supported at the ends and a weight hung from the middle, the rod will gradually sag.

It will be easily seen then that the distinction between solids and viscous fluids is not very marked.

28. Liquids and Gases.—Fluids are again divided into two classes: liquids and gases. A liquid is distinguished from a gas by its having an almost invariable volume. It is very hard to produce mechanically any perceptible difference in the volume of a pound of water, so that for a very long time water was regarded as incompressible.

Gases, however, in their normal condition are easily compressed and are indefinitely expansible, so that they always occupy the whole of any vessel, large or small, in which they may be contained. A gas when highly compressed and near the point of liquefaction is not very different in appearance and properties from a liquid.

29. Vapours.—If a liquid be placed in a closed vessel which it does not entirely fill, the space above it becomes gradually filled with vapour. A vapour is matter in the gaseous condition, but it differs from a gas proper in that if the space which it fills be gradually contracted it passes back to the liquid form. Now a true gas cannot be compressed

by pressure only to a liquid. Carbon dioxide, if below 31° C. (*i.e.* its critical temperature), can be liquefied by pressure and is therefore really a vapour; but oxygen is usually a true gas, for it will not liquefy at any pressure if its temperature is above − 118° C.

30. Mass.—The mass of a body is usually defined as the quantity of matter that it contains. As a matter of fact the term mass is probably better understood than any possible definition of it. If different bodies are acted on by the same force, then their masses are inversely proportional to the accelerations produced on them. This result enables us to compare masses. It is contained implicitly in Newton's Second Law of Motion.

31. Comparison of Masses.—Suppose that we wish to compare the masses of two steel balls, and that we wish to do so without assuming anything beyond what is expressed by Newton's Laws.

Call the balls A and B; their masses m and m'. Let them be hung up side by side as shown in Fig. 14. Behind the balls is placed a mirror scale, SC. Let one of the balls, B say, be drawn aside some definite distance. This distance can be measured on the scale. From it we can find out the vertical distance through which B is raised; call this h. After B is released it will fall and strike A. Its velocity at the instant of impact will be $\sqrt{2gh} = u$ say. After the impact A will move towards the left with some velocity, v' say. The direction in which B moves will depend on the relative masses of A and B. Let us assume that B rebounds from A with a velocity v. The change in velocity of B is $v + u$. The change in velocity of A is v. Hence their changes in momentum are $m(u + v)$ and $m'v'$ respectively.

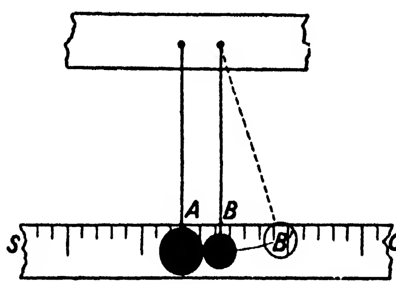


Fig. 14.

Now the forces which have brought about these changes are the action of A on B and the reaction of B on A; by Newton's Law III. these are exactly equal. They have been acting for exactly the same time, and therefore the changes in momentum produced are also equal. Hence

$$m(u+v) = m'v', \qquad \text{or} \qquad \frac{m}{m'} = \frac{v'}{u+v}.$$

The values of u, v, v' may be obtained from measuring the distances (horizontal) which the balls traverse. From these distances the vertical heights to which the balls rise may be calculated, and from these again we get the velocities from the formula $v^2 = 2gh$. (*Vide* Ex. 1, p. 112.)

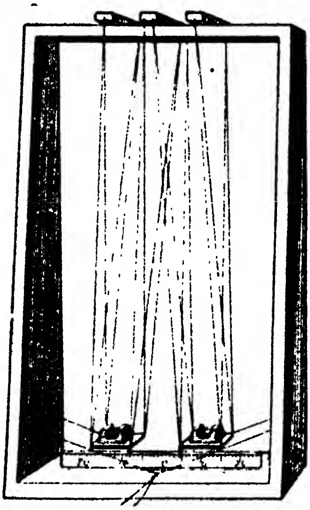


Fig. 15.

32. Hick's Ballistic Balance is shown in Fig. 15. It consists of two platforms arranged to carry the masses to be compared. Each of these carries a pointer which moves over the horizontal scale. By means of a catch, not shown in the figure, the two platforms can be made to stick together after impact.

33. Weight.—The value of the results of Art. 31 is that they afford a fairly direct verification of Newton's Laws. To compare masses by such means would be too troublesome. In practice we make use of the fact that the weights of bodies are proportional to their masses. The simplest proof of this is afforded by the fact that in vacuo all bodies fall at the same rate. Consider two bodies A and B of masses m and m', and weights w and w'. The acceleration of a body is proportional to the ratio of the force acting on it to its mass. The forces causing bodies to fall are their weights, so that $\frac{w}{m}$ and $\frac{w'}{m'}$, are propor-

tional to the accelerations of A and B. These accelerations are identical, since they fall at the same rate.

Hence, $\frac{w}{m} = \frac{w'}{m'}$ i.e. $m : m' :: w : w'$.

The most accurate proof that weight varies as mass is Newton's pendulum experiment. The period of oscillation of a box suspended by a string is the same whether the box is empty or filled with any substance light or heavy.

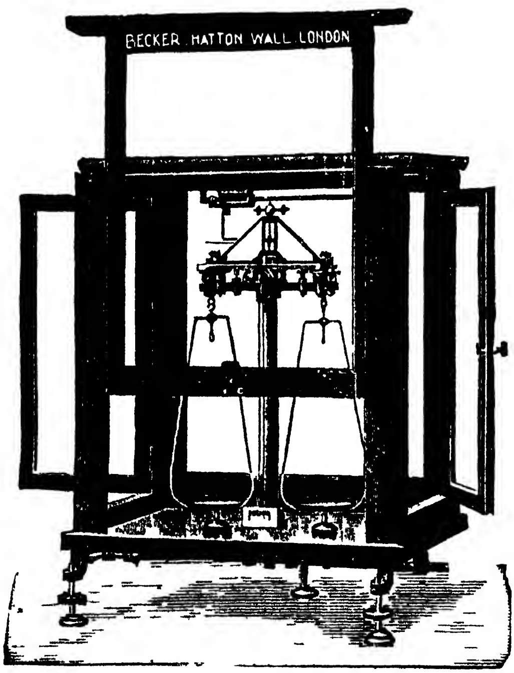


Fig. 16

34. The Balance—This relation between mass and weight enables us to compare masses by means of the balance. The ordinary balance is too well known to need description. Fig. 16 represents an accurate chemical balance.

35. Sensitiveness of a Balance.—Imagine a vertical section is taken through the centre of the beam of a balance, and that this cuts the knife-edges in A, B, C. (Fig. 17.) Let l be the length of each arm, d the depth of the centre of gravity G of the beam below C, and h the height of A and B above C when the balance is at rest and the beam horizontal. Suppose the mass of the beam is M, and that the masses of the scale pans and their loads are m and $m + \mu$, where μ is supposed to be small. The beam will be tilted and inclined to the horizontal at a small angle θ. We are going to find θ.

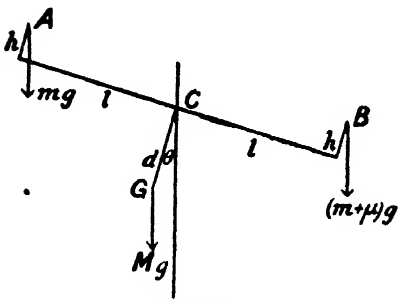


Fig. 17.

The horizontal distance of A from $C = l\cos\theta - h\sin\theta$,

" " " B from $C = l\cos\theta + h\sin\theta$,

" " " G from $C = d\sin\theta$.

Now the angle θ is small, so that we may put $\cos\theta = 1$ and $\sin\theta = \theta$. Take moments about C and we get

$$(m + \mu)g \,.\, (l + h\theta) = Mg \,.\, d\theta + (l - h\theta)mg,$$

$$\therefore\ ml + mh\theta + \mu l = Md\theta + ml - hm\theta$$

if we neglect the product of $\mu\theta$.

Hence, $$\theta = \frac{\mu l}{Md - 2mh}$$

$$\frac{\theta}{\mu} = \frac{l}{Md - 2mh}.$$

This ratio, *i.e.* the ratio of the deflexion to the difference in the loads on the pans, is called the sensitiveness.

Note that if the three knife-edges are coplanar so that $h = 0$, then the sensitiveness is independent of the load.

The sensitiveness is increased by increasing l or decreasing d; the longer the arm of the balance and the nearer the centre of gravity is to the centre knife-edge the greater

the sensitiveness. If the outside knife-edges are above the centre knife-edge, then the sensitiveness increases with the load on the pans; if below, it decreases.

An increase in the mass of the beam decreases the sensitiveness of the balance.

36. Stability.—For work in which great accuracy is required the first essential in a balance is sensitiveness. For many purposes, however, extreme accuracy is not required: the speed with which a weighing may be carried out is often of greater importance: to ensure this the balance must be stable. The stability of a balance is increased by decreasing the length of arm and increasing the depth of the centre of gravity below the point of support of the beam. Stability and sensitiveness are therefore to some extent mutually exclusive.

37. Method of Weighing.—A sensitive balance will swing for so long a time that it is not usual to wait for it to come to rest to see the position the pointer would take up. It is simpler to notice the extent of the swing on either side of the zero of the scale: midway between the two positions is the place which the pointer would occupy if the balance were allowed time to come to rest.

An example will explain the method. Suppose the pointer at the full extent of one swing is opposite to 6 on the left-hand side, that at the end of the next half vibration it is opposite to the 2 on the right-hand side, and swings back to $5\frac{1}{2}$ on the left-hand side again. Take as the extent of the swing $\frac{6+5\frac{1}{2}}{2}$ on the left, 2 on the right: the resting point will be $\frac{1}{2}\left(\frac{6+5\frac{1}{2}}{2}-2\right)$, *i.e.* 1·9 divisions on the left.

Now suppose an extra milligramme added to the weights in the pan brings the resting point to 1·2 divisions on the right. Then 1 milligramme brings the point over 3·1 divisions; therefore the weight that would have brought it to the zero, *i.e.* over 1·9 divisions, is $\frac{1 \cdot 9}{3 \cdot 1}$ milligrammes, *i.e.* ·6 milligramme. The weights originally in the pan are therefore too small by ·6 milligramme. This calculation supposes that the pointer is opposite the zero on the scale when the pans are empty. This must be tested and the true zero found.

38. Faults in Balances.—(1) Scale pans of different masses. To test this interchange the pans. If the pointer reading is altered the pans are of different weights. A small difference in the scale pans does not much matter. The positions of the screws on the arms may be altered to make balance even.

(2) Arms of different lengths. Suppose the lengths are l, l'. Take a body—true mass M, and suppose weights of mass p, p' are required to balance it according as the body is on the right hand or on the left. Then $lp = l'M$, $l'p' = lM$.

From this we get

$$M = \sqrt{pp'}$$

$$\frac{l}{l'} = \sqrt{\frac{p'}{p}},$$

two equations giving both the true mass and also the ratio of the lengths of the arms.

(3) Knife edges. These become rounded by use in the course of time.

(4) Weights untrue. The weights may be to some extent tested by weighing them against one another: thus the 20 gramme weight may be balanced by 10, 5, 2, 2, 1. The only real test, however, is to compare them with weights known to be accurate.

39. Spring Balance.—It is apparent that the ordinary balance shows the equality or inequality of the weights of two bodies. It gives no information as to the weight of any body: it does not indicate the force by which a body is attracted by the earth except in terms of gravitation units. The weight of a body—say the weight of a pound of tea—is greater in London than in Paris, but the ordinary balance cannot show this. At either place the weight of the tea is equal to the weight of a pound at that place. A spring balance, however, measures the weight of a body, because the elongation of the spring is proportional to the force acting on it. Hence at Paris a pound of tea would be shown to weigh less than the same tea in London if a balance sufficiently delicate could be constructed.

EXAMPLES III.

1. Enumerate the states of matter, and mention some properties which do not depend on its state. Give an explanation in each case.

2. Explain by aid of a sketch the construction of a common balance, and state the conditions of equilibrium when it is loaded with unequal weights.

3. State Newton's three laws of motion.

A jet of water is projected against a fixed wall so as to strike it at right angles. If the velocity of the jet be 80 feet per second, and 100 lbs. of water strike the wall in each second, what pressure will be exerted against the wall (*a*) when the water does not rebound, (*b*) when it rebounds with a velocity of 10 feet per second?

4. Distinguish between the mass of a body and its weight. What evidence is there to show that the weights of bodies are proportional to their masses?

5. Upon what qualities of a balance does its sensitiveness depend? How may the sensitiveness of a balance for different loads be determined?

6. A glass ball (Sp. gr. 3) is balanced by brass weights (Sp. gr. 8·4) and is found to weigh 250 grammes. What error does the neglecting of the displacement of the air make if the density of the air at the time of the observation is ·00123 gramme per cubic centimetre?

7. Describe an experiment by means of which the masses of two bodies may be compared without reference to their weights.

8. A force equal to the weight of 10 pounds acts on a mass of 100 pounds for 3 minutes. Find in foot-pound-second units (1) the acceleration, (2) the total work done, (3) the power being expended at the end of each minute.

9. A bullet of mass 20 grammes is shot horizontally from a rifle, the barrel of which is 1 metre long, with velocity 400 metres per second, into a mass of 50 kilogrammes of wood floating on water. If the bullet buries itself in the wood without making splinters or causing the wood to rotate, find the velocity of the wood directly after it is struck (that is, before the velocity has been diminished by the resistance of the water). Also find the average force in grammes-weight exerted on the bullet by the powder ($g. = 981$ cm. per second, per second).

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER IV.

VOLUMES, DENSITY.

40. Volumes of Solids.—The volumes of many solids may be found from their linear dimensions.

Volume of a rectangular block = product of three perpendicular edges.

Volume of a parallelopiped = area of one face × perpendicular distance of that face from the one parallel to it.

Volume of a cylinder = area of base × height.

Volume of pyramid or cone = $\frac{1}{3}$ area of base × altitude.

Volume of a sphere = $\frac{2}{3}$ of volume of circumscribing cylinder ($\frac{4}{3}\pi r^3$).

These results are obtained mathematically.

41. Archimedes' Principle (Chapter XII., Art. 215).—If a body is weighed (i) in vacuo, and (ii) when suspended in water, the difference between the observed weights* is equal to the weight of the water displaced by the body.

If the weight in vacuo = W grammes
" " water = w grammes,

then $W - w$ grammes = weight of water displaced.

Now the weight of one cubic centimetre of water is equal to the weight of a one gramme mass.

$$\therefore \text{ volume of water displaced} = (W - w) \text{ c.c.}$$
$$\therefore \text{ volume of solid} = (W - w) \text{ c.c.}$$

The arrangement of apparatus is shown in Fig. 18, and needs no detailed description.

* In this chapter and elsewhere, where no confusion is likely to arise, the terms mass and weight are used indifferently.

If the body is lighter than water, a sinker is required.
Weigh this in water, w_s grammes.
Weigh body, W grammes.

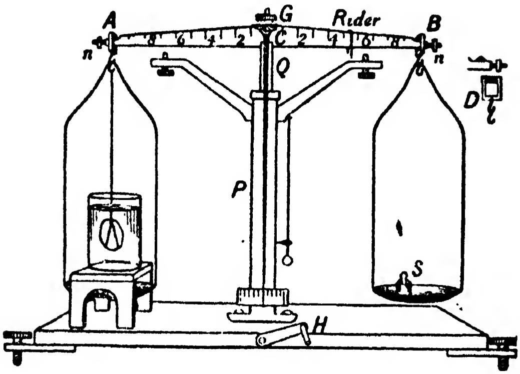


Fig. 18.

Weigh body attached to sinker, both in water, w grammes. The upthrust on the body due to the water $= (W + w_s - w)$ grammes-wt. $\therefore$ volume of solid $= (W + w_s - w)$ c.c.

42. Volumes of Solids soluble in Water.—A liquid, *e.g.* turpentine, ether, alcohol, or a saturated solution of the solid, may generally be found in which the body is insoluble. If ρ grammes per c.c. is the density of such a liquid, W grammes the weight of the solid, w grammes the weight in the solution, then the volume is $(W - w)/\rho$ c.c.

43. Volume of a Powder.—Take the case of sand, shot, or small pieces of glass. Use a specific gravity flask, Fig. 19 (*a*). Suppose the volume of the sand is V c.c.

Weigh the flask, w_1 grammes.

Put the sand in the flask and weigh again, w_2 grammes.

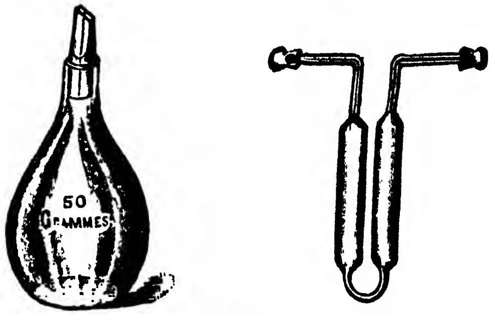
$$\therefore \text{ weight of sand} = (w_2 - w_1) \text{ grammes.}$$

Fill up with water and weigh, w_3 grammes.

Empty out the sand and water, fill up with water and weigh again: w_4 grammes.

If the sand could be poured into the flask filled with

water, the volume of water that would run out would be equal to the volume of sand, i.e. it would be V c.c. and its weight V grammes.



(*a*) Fig. 19. (*b*)

Hence w_3 grammes $+$ V grammes $=$ weight of sand $+$ weight of bottle filled with water $= w_2 - w_1 + w_4$,

$$V = w_2 - w_1 + w_4 - w_3.$$

$$\text{Density} = \frac{w_2 - w_1}{w_2 - w_1 + w_4 - w_3} \text{ grammes per c.c.}$$

44. **Density.**—The density of a substance is defined as its mass per unit volume or the ratio of the mass of any quantity of it to the volume. Its dimensions are ML^{-3}.

45. **Specific Gravity.**—The specific gravity of a substance is the ratio of the mass of any quantity of it to the mass of an equal volume of some standard substance. It has no dimensions and is a mere number.

The standard substance chosen is generally water in the case of solids and liquids. In the case of gases air or hydrogen is sometimes taken as the standard.

In the C.G.S. system in which the unit mass (the gramme) of the standard substance (water) occupies unit volume (the cubic centimetre) the number expressing the density is the specific gravity.

The density of water is 1 gramme per c.c., its specific gravity is 1. The determination of density or specific gravity generally involves calculation of volume.

46. Density of a Liquid.—(1) The usual method is as follows:—

Weigh a s.g. flask, w_1 grammes.

Fill with water and weigh, w_2 grammes.

Empty out the water, dry the flask if necessary, fill with liquid and weigh again, w_3 grammes.

The weight of water $= (w_2 - w_1)$ grammes.

The volume of the flask and therefore of the liquid filling it $= (w_2 - w_1)$ c.c.

$$\therefore \text{density of liquid} = \frac{w_3 - w_1}{w_2 - w_1}.$$

Fig. 15 (*b*) shows a convenient form of flask for liquids which can only be obtained in small quantities.

(2) Weigh a solid (1) in vacuo, w_1 grammes,
(2) in water, w_2 grammes,
(3) in the liquid, w_3 grammes.

Weight of water displaced $= (w_1 - w_2)$ grammes.

Weight of liquid displaced $= (w_1 - w_3)$ grammes.

$$\text{Hence specific gravity of liquid} = \frac{w_1 - w_3}{w_1 - w_2}.$$

A refinement of this method can be used when it is required to find how the density of water changes with the temperature.

47. Density of Gases.—In the case of air, weigh a flask of known volume, and then exhaust and weigh again. A very simple method which does not require an air pump is to take a round-bottomed flask fitted with a cork. Through the cork passes a glass tube the end of which is inserted into a piece of rubber tubing. A screw pinch cock is fitted on the rubber. If a little water is put in the flask and boiled briskly, the air in the flask will be driven out by the steam. Close the flask while still boiling; weigh. Open the flask and the air enters. Weigh again. The difference in the weighings divided by the volume of the flask gives the difference in densities of air and saturated water vapour.

Other methods are dealt with in books on Physical Chemistry.

48. U Tube.—This method is useful for finding the specific gravity of a liquid which does not mix with water.

Take a wide U tube, fill nearly half full with water. Pour into one arm the liquid, say turpentine, the specific gravity of which is required.

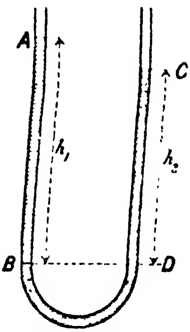


Fig. 20.

Suppose that A, C (Fig. 20) are the free surfaces of turpentine and water, B the surface of separation.

If h_1, h_2 are the heights of A and C above B, ρ the specific gravity of turpentine, P the atmospheric pressure, then the pressure at $B = P + g\rho h_1$ (Art. 210).

But pressure at B = pressure at D at same level in same liquid

$$= P + g h_2$$

$$\therefore \; g h_2 = g h_1 \rho$$

$$\rho = \frac{h_2}{h_1}$$

To find h_1, h_2 it is generally convenient to measure the heights of A, B, and C from the bench.

49. Hare's Apparatus.—If two liquids mix, Hare's apparatus is convenient for comparing their densities (Fig. 21). The pressure in the two prongs of the tube is diminished by sucking air out at the top. The liquids then rise to different heights in the two prongs.

The pressure at A = pressure at B,

the pressure at C = atmospheric pressure

= pressure at D,

$g\,\rho_1\,h_1$ = pressure difference between A and C

= pressure difference between B and D

$= g\,\rho_2\,h_2$

$$\therefore \frac{\rho_1}{\rho_2} = \frac{h_2}{h_1}.$$

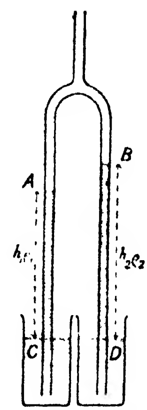


Fig. 21.

50. Hydrometers.—Unless great accuracy in determining the density of a liquid is required, the common hydrometer is usually employed.

In Figure 22 A is a glass stem, B a hollow bulb, C a bulb filled with mercury. The instrument is allowed to float in the liquid the density of which is required.

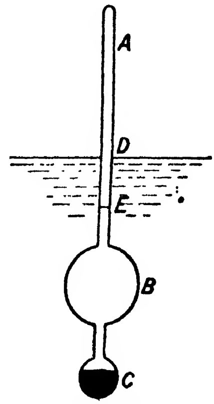


Fig. 22.

Suppose a = sectional area of stem,
D is the level at which it floats in water,
E is the level at which it floats in a liquid of density ρ.
V = volume of hydrometer below D, *i.e.* volume of water displaced.

$$DE = h.$$

The mass of the liquid displaced must always be equal to the mass of the instrument.

The volume of liquid displaced is less by ah than the volume of water displaced.

$$\therefore (V - ah)\rho = V$$

$$\therefore h = \frac{V(\rho - 1)}{a\rho}.$$

By means of this relation the instrument can be graduated.

51. Volumenometer.—This is used in cases in which none of the preceding methods avail: *e.g.* for a powder which cannot be immersed in water or other liquid.

The volumenometer (Fig. 23) is a glass vessel, A, connected by a glass tube BC to the wider tube CD. The latter is joined by a rubber tube EF to a glass vessel K. Two marks are placed at C and D. The volume of the tube between them must be known: call it v.

The stopper G is air-tight. To use the instrument, the substance the density or volume of which is required is placed in the vessel A. Mercury is poured into K till it reaches the level D. The stopper G is fitted. K is then raised till the mercury rises to C. The height of the

mercury in K above C is noted. This gives the pressure p to which the air in the vessel A is subjected.

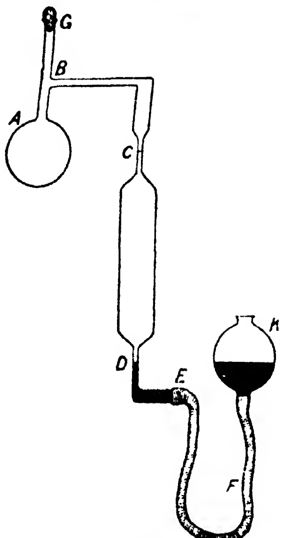


Fig. 23.

Suppose x = volume of substance,

V = volume of A and the tube BC,

P = atmospheric pressure.

The volume of the air at first contained in the instrument is $V - x + v$.

This was at a pressure P.

Its final volume was $V - x$, and the pressure $P + p$.

Hence by Boyle's Law (Art. 170).

$$(V - x)(P + p) = (V - x + v) P$$

$$\therefore x = V - P\frac{v}{p}.$$

This equation determines the volume of a substance in terms of known or measurable quantities.

Examples IV.

1. Define the terms density, specific gravity. In what circumstances do the numbers expressing these two values coincide?

2. Find the specific gravity of a solid substance from the following data:—

A flask, which when filled with water weighs altogether 410 grammes, has 80 grammes of a solid introduced, and being then filled up with water weighs 470 grammes. What is the volume of a kilogramme of the solid?

3. A piece of lead and a piece of sulphur are suspended by fine strings from the extremities of a balance beam and just balance each other in water. Compare their volumes, their densities being respectively 11·4 and 2 grammes per cubic centimetre. Which of them will appear to be the lighter in air, and what weight must be added to it to restore equilibrium?

4. Two pieces of iron (Sp. gr. 7·7) suspended from the two scale-pans of a balance, the one in water and the other in alcohol of Sp. gr. 0·85, are found to weigh exactly alike. Find the proportion between their true weights.

5. A solid, of which the volume is 1·6 cubic centimetres, weighs 3·4 grammes in a fluid of specific gravity 0·85.

Find the specific gravity and weight of the substance.

6. A beaker containing water rests on a table. A lump of iron (Sp. gr. = 7·5) weighing 500 grammes is hung from a piece of cotton. If the lump be lowered into the beaker, how will the tension of the cotton change? Will the thrust of the beaker on the table be altered?

7. A lump of ice floats in a beaker which is just full of water. What will happen to the level of the liquid when the ice melts?

8. A cork is floating on the water which fills a measuring cylinder up to the 50 c.c. mark. If a 10 gramme weight be placed on the cork, what will be the reading of the water level?

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER V.

ENERGY.

52. It is perhaps not too much to say that of all the discoveries that have been made during the past century the greatest and most important is that of the doctrine of the Conservation of Energy. We propose in this chapter to consider this doctrine, to examine the evidence for it, and to trace out some of its consequences. We shall begin with elementary mechanics.

53. **Work** is generally defined as follows: A force is said to do work when the point of application of the force moves in the direction of the force: it is measured by the product of the magnitude of the force and the distance moved in the direction of the force. In other words work is force multiplied by distance. Its dimensions are therefore $M^1L^2T^{-2}$.

Suppose a body move from a point A along the straight line AB to the point B, and that it is acted upon during the whole time of its motion by a constant force, F, in the direction AX. Draw BN perpendicular to AX. Then the distance which the body has moved in the direction AX is AN. The work that the force F has done is the product $F.AN$. Now $AN = AB\cos A$; hence the work $= F.AB\cos A = F\cos A.AB =$ resolved part of F in the direction AB multiplied by AB. This definition of work as the product of the distance moved and the resolved part of the force is often useful.

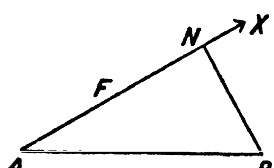


Fig. 24.

54. Units of Work.—The unit in which it is usual to measure work is the foot-pound. A foot-pound is the work done by a force equal to the weight of one pound when the body on which it acts moves one foot in the direction of the force. It is the work done in lifting a mass of one pound through a distance of one foot. This unit of work, though very convenient for engineers, is a gravitational unit; hence it is not constant at different places. For scientific purposes it is usual to state results in absolute units. The absolute unit on the C.G.S. system is the erg, the work done when a force of one dyne moves its point of application over a distance of one centimetre in the direction of the force. The Joule is a multiple of this, and is equal to 10^7 ergs.

55. Energy.—Now any body or any system of bodies which is capable of doing work is said to have energy. Energy is therefore the capacity for doing work. As examples of bodies having energy take a watch spring, which when wound can drive the mechanism of a watch against friction many hours. Again, consider a stream of water just before it enters a turbine or undershot water-wheel; the energy of the water is passed on to the turbine, and the water leaves with a much reduced velocity. These two examples illustrate different forms of energy. The watch spring had energy in virtue of its strained condition, the water in virtue of its motion. The one kind of energy is termed Potential energy, the latter Kinetic energy.

The energy of a system is measured by the amount of work that it can do. It is thus of the same nature as work, and has the same dimensions.

56. Measure of Kinetic Energy.—Now let us find the work that a body of mass m, say a stone, can do if it is moving with a velocity v. Let a force F act upon it which brings it to rest after it has travelled a distance s. The work done on it in stopping it, *i.e.* the work that it can do in virtue of its motion, is Fs. While the force F acts on the stone it causes a constant retardation of F/m. Call

this a. Apply the formula $v^2 = u^2 + 2as$. The final velocity of the stone is 0; therefore $v^2 = 2as = 2 \cdot \frac{F}{m} s$. Hence the work done in stopping the stone $= Fs = \frac{1}{2}mv^2$. Hence the kinetic energy of a moving body is half the product of the mass and the square of the velocity. (See, however, Art. 78.)

57. The total energy of a falling body is the same at all points.—Suppose the stone considered in the previous article has fallen a distance x. Its kinetic energy $= \frac{1}{2}mv^2 = mgx$, since $v^2 = 2gx$. It still has a distance $h - x$ to fall, so that its potential energy is $mg(h - x)$, for it is acted on by its weight—a force equal to mg—which can cause it to fall a distance $h - x$. Hence the total energy $= \mathrm{P.E} + \mathrm{K.E} = mgh$. This is therefore constant, and equal to the energy originally possessed by the stone.

The falling stone affords a very simple example of the conservation of energy. It is important to notice that the result was obtained on the assumption that the stone fell in vacuo. When a stone falls through air its motion is impeded by frictional forces, so that part of the work done by gravity goes to warming up the air and stone. The total energy, however, still remains constant, but the energy is not entirely the energy of motion of gross matter.

58. Perpetual Motion.—One of the problems that philosophers of the middle ages vainly attempted to solve was the invention of a machine capable of producing perpetual motion. For perpetual motion there is required not merely a system which will go on moving for ever when once started, but one that will continue to do external work. If the energy of a system is a constant quantity, then such a motion is evidently an impossibility. It may be worth while, however, to glance at a device which was very common in some shape or other and appears in most machines designed to produce perpetual motion.

Suppose a smooth heavy rope, ABC, lies on an inclined plane as shown in the figure, passing over a pulley at B.

What will happen? Will the short length BC pull the longer part AB up the plane, or will AB pull BC over? Imagine that BC is capable of pulling AB up. Next suppose that C and A are joined together with another length of smooth rope and that this lies in a horizontal channel. Then if BC pulls AB up, the rope CA will follow round, so that the ring of rope will go on moving faster and faster and perpetual motion is established.

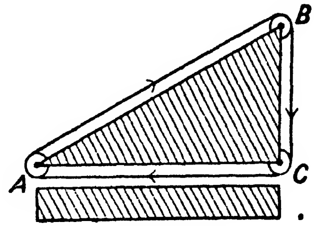


Fig. 25.

Of course as a matter of fact the part BC balances and just balances the part AB, so that were such a frictionless system set going, it would continue to go with a uniform motion.

59. The Inclined Plane: Machines.—We can now deduce the mechanical advantage of the smooth inclined plane: for we have

$$\frac{\text{weight of } BC}{\text{weight of } AB} = \frac{\text{length of } BC}{\text{length of } AB} = \frac{\text{height of plane}}{\text{length of plane}},$$

so that if a weight W rest on an inclined plane the force P which, acting parallel to the plane, is required to support it is given by the relation

$$\frac{P}{W} = \frac{\text{height of plane}}{\text{length of plane}},$$

and the work done by P in any displacement = work done against gravity in lifting the weight.

It was by this line of reasoning that Stevin discovered the laws of the inclined plane.

In any machine the ratio of the "resistance" (or "weight") to the "effort" (or "power (?)") is termed the *mechanical advantage*. This ratio depends on the nature of the machine, its frictional resistance, etc. The ratio of the distance traversed by the point of application of the "effort" to the corresponding distance traversed by the point of application of the "resistance" is termed the *velocity ratio*. It is dependent only on the geometrical construction

of the machine. The ratio of the work done against the "resistance" to the work done by the "effort" is termed the *efficiency*.

In any *frictionless* machine the mechanical advantage and the velocity ratio must be equal, and the efficiency is unity. In every actual machine the mechanical advantage is less than the velocity ratio, and the efficiency is less than unity.

In the case of ordinary machines (*e.g.* lever, pulley, screw, etc.) the principle of conservation of energy enables us to obtain the usual relations between the "effort" and the "resistance" if we know the relations between the distances traversed by the points of application of these forces in any motion that takes place: for the ratio of the forces in the case of frictionless machines must be the inverse ratio of the distances: the work done by the effort must be equal to the work done against the resistance. Were this not the case one machine working forward could rather more than drive another working backward, and there would be a self-acting engine doing external work: *i.e.* perpetual motion would be established.

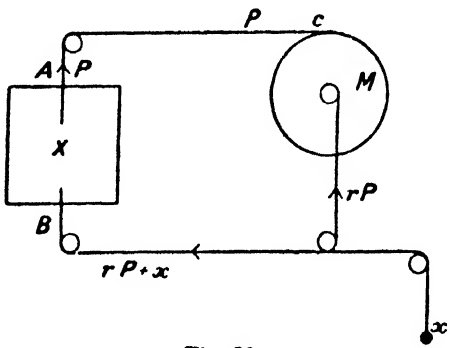


Fig. 26.

Suppose for instance that we have a frictionless machine X, Fig. 26, in which a resistance W can be supported by an effort P: call the velocity ratio r: then $W = rP$. To show this let us imagine $W = rP + x$.

Couple up W and P to a wheel and axle (M) of velocity ratio r: we shall assume that in this machine we know from experiment or otherwise that the mechanical advantage is also equal to r. We may imagine that in X one foot of rope at A is drawn out under a

pull P: this must cause $\frac{1}{r}$ foot of rope to be drawn in at B under a pull $rP + x$. This pull will serve to draw up a weight (x) and to drive the wheel and axle backwards, causing the rope at C to move a distance of one foot under a pull P. The combined machine M and X is therefore self-acting and does external work in lifting x.

One step further and we get the general result that under any conservative system of forces the total work done in producing any change is constant and independent of the way in which that change takes place.

(In a conservative system the magnitude and direction of the forces acting do not depend on the direction in which motion takes place. In mechanics a conservative system is practically one in which there is no friction.)

In every real machine frictional forces come into play: consequently the energy absorbed is always greater than the energy obtained from it. The difference between the two amounts is the energy of the heat developed. Hence the doctrine of the conservation of energy could not possibly be formulated as long as heat was regarded not as the energy of molecular motion, but as a material substance—caloric. The experiments of Rumford showed that heat was produced by work: that heat was energy.

60. **Mechanical Equivalent of Heat.**—It remained for Joule to obtain quantitative relations between the amount of work done and the heat produced. It was by showing that a certain definite quantity of work always corresponded to another definite quantity of heat,—or, in other words, that when a definite amount of energy of any kind disappeared there was then always developed a definite amount of heat,—that he was enabled to establish the doctrine on the firm basis of experiment. The mechanical equivalent of heat is the amount of energy measured in mechanical units, that is equal to a unit of heat. For the methods by which Joule found this equivalent, and for the account of the experiments of other physicists, we must refer to books on heat. It will, however, probably be of interest to give a few of the results obtained by different

methods. These are extracted from a list in Preston's *Heat*:—

Date.	Observer.	Method.	Result in kilogramme-metres.
1843.	Joule.	Friction of water in tubes.	424·6.
1850.	Joule.	Friction of mercury.	424·7.
1858.	Hirn.	Friction of metals.	400-450.
1860.	Hirn.	Expansion of air.	440.
1879.	Rowland.	Friction of water between 5° and 36°.	429·8-425·8.
1842.	Mayer.	By relation $J = \frac{p_0 v_0 a}{C_p - C_v}$ for gases.	365.
1867.	Joule.	Heating of a wire by electric current.	429·5.
1893.	Griffiths.	,, ,, ,, ,,	427·45.
1899.	Callendar and Barnes.	Heating of a mercury column by an electric current.	426·52.

61. The First Law of Thermodynamics.—The values obtained by so many different methods have so close an agreement that even if there were no other evidence we could regard the first law of Thermodynamics as established. This law may be stated thus: When mechanical work is converted into heat or heat into work, then the ratio of the work to the heat is constant. This constant is the mechanical equivalent of heat.

62. Conservation of Energy.—Clerk Maxwell's statement of the Principle of the Conservation of Energy runs as follows:—

"The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible." (*Matter and Motion.*)

The most convincing proof of the principle is probably the fact that very many other principles are founded on it, and that these principles stand the test of experiment.

63. Dissipation of Energy.—Though it is convenient to talk of frictionless machines, yet in actual practice all machines work with more or less loss of mechanical energy: the balance appears as heat. Whenever work is done some energy is spent in heat: this energy is eventually lost to all intents and purposes. It would further appear that

energy of all forms must finally degrade to heat, though it may pass through many stages before it does so. Take the energy of sound: this is energy due to strain and motion in the air. Generally this energy disappears at once as heat: the air is slightly warmed by sound passing through it. The energy may, however, be caught by a telephone and converted into the energy associated with an electric current: but this current warms the wire and produces motion in the receiver, so that the sound energy is still dissipated as heat.

64. **Available Energy.**—Now from high temperature heat we can obtain mechanical energy. This is the most usual way of driving machinery and obtaining work. Low temperature heat on the other hand cannot be employed usefully. The heat energy in the atmosphere or in the ice of the Arctic regions is vastly greater than the heat energy that can be obtained by burning a ton of coal. But it is only the coal that is serviceable. It has often been stated that the heat of the atmosphere can be utilised by means of liquid air: and so it can be, given the liquid air, but the heat of the atmosphere is in this case high temperature heat. The liquid air is at a far lower temperature. Air is liquefied by mechanical energy. A pint of liquid air might then be used to drive a liquefier and produce other liquid air, but the amount of this must be less than the pint employed to make it. This follows from the principle—known as the Second Law of Thermodynamics—that heat cannot be made to pass from a body at low temperature to one at high temperature, except by the expenditure of work. The heat of the atmosphere is therefore not an available source of energy.

65. **Sources of Energy.**—The sources of energy which are practically available are very few. The most important are—

(1) Chemical affinity.
(2) Waterpower.
(3) Wind.

(1) The chemical affinity of carbon and hydrogen for oxygen is at present the chief method of supplying power

in England. The burning of coal or coal gas evaporates water to steam; the internal combustion engines burning petrol or alcohol supply power more directly. We have also a small source of power obtained by burning such things as sulphur, gunpowder.

(2) Waterfalls have long been usefully employed to drive flour mills. In hilly districts waterfalls are often employed to drive turbines actuating dynamos. These supply power at places which are often very distant from the fall.

(3) Wind-mills are somewhat clumsy and seem to need attention, though they are much used in flat countries.

It is interesting to note that all or almost all energy now available has been derived at some time or other from the sun. Plants are enabled by means of the green pigment chlorophyll to absorb energy from the sun's rays. Some of this energy is available for preparing the food of the plant out of carbon dioxide, water, and salts from the soil. The food not immediately needed is stored away in seeds, leaves, etc.

Animals feed upon this stored energy, and man upon animals and plants, so that it is by virtue of solar energy that men do their work.

Besides this, coal is but the remains of vegetation buried ages ago; peat and lignite are less carbonised remains. Alcohol is a product of the fermentation of starch and sugar. Petroleums are thought to be decomposition products of animal and vegetable remains.

Water is carried up from the sea as vapour and condensed on hilltops, whence it runs down supplying the energy of waterfalls. The evaporation is caused by the heat of the sun.

Winds are convection currents caused by temperature difference. These again are due to the sun. Solar heat has been employed experimentally on a small scale, the rays being concentrated on the boiler of a steam engine.

There are other forms of energy of minor importance: thus the tides in a few instances drive tidal engines. The energy of the tides is due not to the sun nor to the moon, but to the earth. The tides, held by the moon, act

as a brake on the earth and are continuously slowing it down, thus making its period of revolution larger. The earth once caused tides on the moon. The brake thus applied to the moon has been sufficient to stop the revolution of the moon relative to the earth, so that the moon always turns the same face towards us.

The internal heat of the earth is another source of energy which may some day be utilised.

Whence the sun gets its energy is not definitely known. Till recent years it has been generally held to be due to contraction under the action of gravity. Doubtless much of it is due to that cause. Possibly, however, actions such as the degradation of radium play a part in keeping up its temperature.

EXAMPLES V.

1. Discuss the practicability of using tidal energy as a source of power on a large scale. What is the origin of tidal energy?

2. A stream of water, flowing at the rate of 1100 lbs. / sec., turns an undershot water-wheel. It comes against the paddles with a velocity of 8 ft. / sec., and leaves with the velocity of the paddles 4 ft. / sec. Find the rate of loss of momentum and the rate of working of the wheel.

3. Assuming as the result of experience the impossibility of perpetual motion, deduce the law of the inclined plane.

4. Explain where the energy goes when you expend it in (*a*) winding up a watch, (*b*) lifting a stone from the floor and placing it on a shelf, (*c*) riding a bicycle up-hill, (*d*) rowing a boat on a still pond, (*e*) rowing a boat up-stream.

5. Prove that the energy of a moving body is proportional to the square of its velocity.

6. Explain the terms Conservation and Transformation of Energy. Illustrate your answer by reference to an electric light circuit, a gas engine being used to drive the dynamo.

7. Show how to find the energy stored in a stretched elastic spring.

8. Find the work done in compressing a gas at a pressure p and volume v to volume $v/2$. Assume that the gas obeys Boyle's Law.

9. What relation exists between the efficiency, velocity ratio, mechanical advantage in any machine?

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER VI.

CIRCULAR MOTION.

66. Change of Velocity.—Suppose a body P moves in a direction AB (Fig. 27a) with a velocity u. If a force acts on P, its velocity will gradually change. Suppose that the velocity change to a velocity v in the direction AC. Now the change in velocity of P is not $u - v$, for velocity is a vector quantity, and vector quantities, unless parallel, cannot be directly added or subtracted; the change in velocity is that velocity which combined with the original velocity u has a resultant velocity v. The easiest way of finding this velocity is to draw lines Ab, Ac (Fig. 27b) to represent u and v and to join bc. Then bc represents the change in velocity.

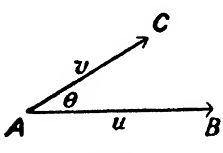


Fig. 27a.

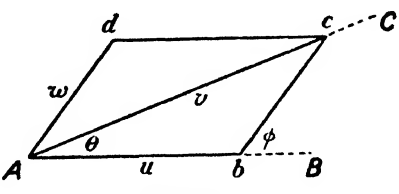


Fig. 27b.

To show this complete the parallelogram $Abcd$. Then the velocity represented by bc is the velocity represented by Ad, and the parallelogram of vectors (Art. 9) tells us that velocities represented by Ab, Ad are together equivalent to a single velocity represented by Ac. Denote by w the velocity represented by bc. Then the velocities u, w are together equivalent to v. In other words, w is the change in velocity that the body P has undergone.

Denote the angles BAC, Bbc by the symbols θ, ϕ. The change in velocity w may be resolved into the two components $w \cos \phi$ along AB, $w \sin \phi$ at right angles to AB.

$$w \cos \phi = v \cos \theta - u$$
$$w \sin \phi = v \sin \theta.$$

Hence the change in velocity along $AB = v \cos \theta - u$, and the change perpendicular to $AB = v \sin \theta$.

Let us suppose that the change in velocity has been brought about gradually and uniformly, and that it has occupied a time t: then the acceleration in the direction AB has been $\dfrac{v \cos \theta - u}{t}$, for acceleration is defined as the rate of change in velocity, *i.e.* the change in velocity divided by the time. Similarly the acceleration perpendicular to $AB = \dfrac{v \sin \theta}{t}$.

67. Hodograph.—Suppose a body P move along the curve ABC, and that its velocity at A is v_1, at B is v_2, and so on. (Fig. 28.)

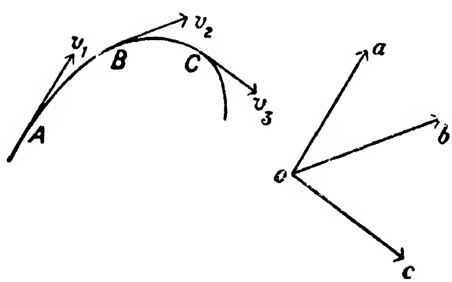


Fig. 28.

Take any point o and from it draw lines oa, ob, ... to represent v_1, v_2, ... When the points A, B, C, ... are taken very close together, and the points a, b, c ... are joined up by a continuous line, this continuous line is termed the hodograph for the motion of P. Take a few simple examples.

(1) If the point P moves in a straight line with uniform velocity, the points a, b, c, . . . will all be in the same place, so that the hodograph is a single point.

(2) If P moves in a straight line with variable velocity, the case of a stone falling vertically, the lines oa, ob, . . . will all be drawn in the same direction, but will be of different lengths, so that the hodograph is a vertical straight line passing through o.

(3) In the case of a stone projected with some horizontal velocity the path described is a parabola. The horizontal velocity is always constant and equal to the initial horizontal velocity, so that the points a, b, . . . on the hodograph are always the same horizontal distance from o, for the projection of the lines oa, ob, etc., on the horizontal represent the horizontal components of the velocities v_1, v_2, etc. The hodograph is therefore a vertical straight line which does not pass through o.

(4) If P moves in a circle with uniform speed (u), the hodograph of the motion will be a circle, for all the lines oa, ob, etc., are of the same length (u), but are drawn in different directions. If the circle were described with variable speed, the hodograph might be an oval curve surrounding o.

68. **The Velocity in the Hodograph.**—Suppose that A and B are points close together on the path of P. Then a and b will be close together on the hodograph, and the angle aob will be small. (Fig. 29.)

Suppose P moves from A to B in time τ, and that the velocity changes from u at A to v at B. Imagine that, while P describes the curve ABC . . ., a point p describes the hodograph abc . . . . Then p moves from a to b in time τ, so that its velocity is $\dfrac{ab}{\tau}$.

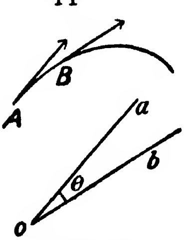


Fig. 29.

Denote by θ the angle aob.

Now since oa represents the velocity at A, and ob represents the velocity at B, it follows from Art. 66 that ab represents the change in velocity that P undergoes in passing from A to B.

The rate of change of velocity of P is represented by $\frac{ab}{\tau}$, *i.e.* by the velocity of p in the hodograph.

Hence at any instant the acceleration of P is represented by the velocity of p in the hodograph.

39. **Uniform Circular Motion.**—Suppose P moves in a circle, centre O, radius r, with uniform speed v. (Fig. 30.)

Then, adopting the notation of Art. 67, $oa = ob$.

Now the velocity of P at any instant is always at right angles to the radius drawn through P: hence oa, ob are perpendicular to OA, OB, and the angle $AOB = aob = \theta$ (measured in circular measure).

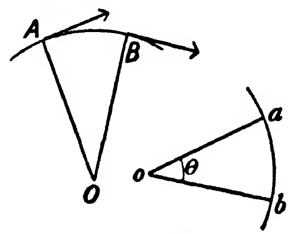


Fig. 30.

The small arc AB is described in time τ,

$$\therefore\ v = \frac{\text{arc } AB}{\tau} = \frac{r\theta}{\tau}.$$

The velocity of p in the hodograph $= \frac{ab}{\tau} = \frac{v\theta}{\tau}$,

$$\therefore \text{ acceleration of } P = \frac{v\theta}{\tau} = \frac{v\theta v}{r\theta} = \frac{v^2}{r}.$$

Now ab is small and is therefore in the limit perpendicular to oa, *i.e.* parallel to AO.

Hence the acceleration of P is $\frac{v^2}{r}$ and is always directed along the radius.

As P describes the circle at a constant rate, the radius joining O to P rotates at a constant rate and sweeps out equal angles in equal times. If ω is the circular measure of the angle described by the radius in unit time, ω is called the angular velocity of P about O.

The circumference of the circle is of length $2\pi r$, P moves round this in time $\frac{2\pi r}{v}$. The angle described by OP in a

complete revolution is 2π, so that the time of describing it must be $\frac{2\pi}{\omega}$;

$$\therefore \frac{2\pi r}{v} = \frac{2\pi}{\omega},$$

$$i.e.\ v = r\omega.$$

Hence acceleration of $P = \frac{v^2}{r} = r\omega^2$.

70. Motion in a Circle (Aliter).—Suppose a particle describes a circle of radius a with uniform velocity v.

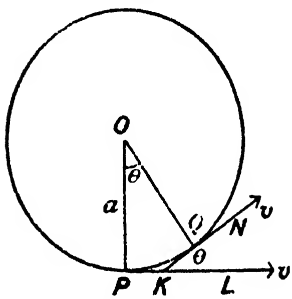


Fig. 31.

Let P be its position at any instant, and suppose that in a short time, t, it has moved on to Q. Join the centre O to the points P, Q. Let the tangents PKL, KQN at P and Q meet in K.

Since the angles at P, Q are right, the angle LKN must be equal to the angle POQ. Call this angle θ.

At the beginning of the time considered the particle is moving with a velocity v in the direction PL; at the end of the time t it is moving in a direction QN, inclined to the former direction at an angle θ. The velocity has therefore changed, and the particle must have some acceleration. We shall find the components of this acceleration in the directions PL, PO.

The component of the velocity at P in the direction $PL = v$.

The component of the velocity at Q in the direction PL

$$= v \cos NKL = v \cos \theta.$$

Change of velocity in the direction $PL = v \cos\theta - v$

$$= v(\cos\theta - 1)$$

$$= v\left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \ldots - 1\right)$$

$\therefore$ rate of change of velocity in the direction PL, *i.e.* the acceleration in this direction,

$$= \frac{v}{t}\left(-\frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \ldots\right)$$

$$= \frac{v\theta^2}{t}\left(-\frac{1}{\underline{|2}} + \frac{\theta^2}{\underline{|4}} - \ldots\right).$$

Now the arc PQ ($= a\theta$) is described in time t,

$$\therefore\ vt = a\theta,$$

$$\therefore\ \frac{v\theta}{t} = \frac{v^2}{a}.$$

Hence the acceleration in the direction PL

$$= \frac{v^2}{a} \cdot \theta\left(-\frac{1}{\underline{|2}} + \frac{\theta^2}{\underline{|4}} - \ldots\right).$$

Now the smaller θ is, the more nearly does this expression approach zero, so that taking the limiting case in which $\theta = 0$, we find that the acceleration in the direction of the tangent is zero.

Again, the component of the velocity at P in the direction $PO = 0$, the component of the velocity at Q in the direction $PO = v \sin NKL$

$$= v \sin \theta,$$

$\therefore$ change of velocity in the direction $PO = v \sin \theta$,

hence rate of change of velocity in this direction, *i.e.* the acceleration

$$= \frac{v \sin \theta}{t}$$

$$= \frac{v\left(\theta - \frac{\theta^3}{\underline{|3}} + \ldots\right)}{t}$$

$$= \frac{v\theta}{t}\left(1 - \frac{\theta^2}{\underline{|3}} + \ldots\right)$$

$$= \frac{v^2}{a}\left(1 - \frac{\theta^2}{\underline{|3}} + \ldots\right)$$

$$= \frac{v^2}{a}$$

in the limit when $\theta = 0$.

Hence the acceleration of the particle is directed wholly along the radius of the circle and its magnitude is $\frac{v^2}{a}$.

If ω is the angular velocity of the particle, then $v = a\omega$, and the acceleration is equal to $a\omega^2$.

71. Force acting on a Body moving in a Circle.—The fact that a body moving with uniform angular velocity in a circle has an acceleration towards the centre implies the existence of some force in the same direction producing that acceleration; thus if m be the mass of the particle considered in Articles 69, 70, the force acting on it will be

$$\frac{mv^2}{a} = ma\omega^2.$$

This force may be exerted in different ways; in the case of a stone fastened to a string we have the tension of the string, in the case of the planets it is gravitation which prevents them from travelling off into space along tangents to their paths.

Consider the case of a stone whirled round at the end of a string the other end of which is held by the hand. The stone is acted on by a force acting always along the string towards the hand, *i.e.* by a "centripetal" force. The force which the hand experiences is, in accordance with Newton's Third Law, exactly equal and opposite to the force acting on the stone. This force is sometimes termed the "centrifugal" force.

72. Conical Pendulum.—Suppose a particle is supported by a string and moves round with constant velocity in a circle, while the string by which it is attached to a fixed point traces out a cone. The particle and string constitute a simple conical pendulum.

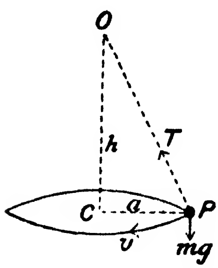


Fig. 32.

Suppose P is the particle, m its mass, v its velocity, a the radius of the circle it describes. Then the acceleration of P is $\frac{v^2}{a}$, and is directed towards the centre.

Call O the fixed point, and C the centre of the horizontal circle it describes. Let $OC = h$.

The forces acting on the particle are

(1) T, the tension of the string in the direction PO;

(2) its weight, mg, acting vertically downwards.

These forces together produce an acceleration $\frac{v^2}{a}$ in the direction PC, and their resultant must therefore be of magnitude $\frac{mv^2}{a}$ and in the direction PC.

Hence resolving (1) along PC and (2) at right angles to PC we get

$$T' \cos OPC = \frac{mv^2}{a},$$

$$T' \sin OPC = mg.$$

Eliminating T by division and calling the semi-vertical angle of the cone α and the length of the string l, these equations give

$$v^2 = ag \tan \alpha = \frac{a^2 g}{h}.$$

If ω is the angular velocity of P round C,

$$\text{then } a\omega = v \text{ and we get } \omega^2 = \frac{g}{h} = \frac{g}{l \cos \alpha}.$$

73. Rod describing a Cone.—Take the case of a uniform rod, length l, mass M. Let this revolve round the vertical OC with uniform angular velocity ω, describing a cone of angle α.

We are going to find a relation between α and ω. Is the motion of the rod the same as the motion of a simple conical pendulum of length $\frac{l}{2}$, *i.e.* is the motion of the rod the same as if its whole mass were concentrated at its centre of mass? In that case we shall find $\omega^2 = 2 \frac{g}{l \cos \alpha}$.

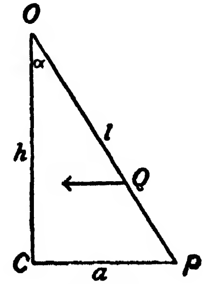


Fig. 33.

Suppose the rod is divided up into a very large number of portions of masses $m_1, m_2, m_3, \ldots\ldots$, and that the distances of these from O are

$$r_1, r_2, r_3, \ldots\ldots.$$

Consider one of them, Q say, of mass m_s, distance r_s.

This describes a circle radius $r_s \sin \alpha$ with angular velocity ω. The force required to produce such an effect is $m_s . r_s \sin \alpha . \omega^2$, and its moment about O is

$$m_s . r_s \sin \alpha\, \omega^2 r_s \cos \alpha = \omega^2 . r_s^2 m_s \cos \alpha \sin \alpha.$$

The sum of the moments of all such forces about O

$$= (m_1r_1^2 + m_2r_2^2 \ldots\ldots)\,\omega^2 \sin\alpha\cos\alpha$$
$$= \omega^2 \sin\alpha \cos\alpha\, \Sigma mr^2.$$

Now the only external force acting on the rod which has a moment round O is the weight of the rod. Its moment round O is $Mg\dfrac{l}{2}\sin\alpha$.

$$\therefore\ \omega^2 \sin\alpha\cos\alpha\,\Sigma mr^2 = Mg\frac{l}{2}\sin\alpha,$$

$$\therefore\ \omega^2 = \frac{Mgl}{2\cos\alpha\,\Sigma mr^2}.$$

Now $$\Sigma mr^2 = M\frac{l^2}{3}\ \text{(see Art. 74)},$$

$$\omega^2 = \frac{3}{2}\frac{g}{l\cos\alpha}.$$

This result is interesting because the motion of the rod is the same as if its mass were concentrated not at the centre, but two-thirds of the way down.

74. Moment of Inertia of a Rod about one End.—In Art. 73 we used the result

$$\Sigma mr^2 = M\frac{l^2}{3}.$$

We now propose to prove it.

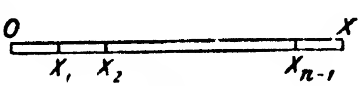


Fig. 34.

Take a thin rod OX, of length l, mass M, and line density (*i.e.* mass per unit length) ρ, then

$$\frac{M}{l} = \rho.$$

Divide the rod into n equal parts at the points

$$X_1,\ X_2,\ X_3,\ \ldots\ldots,\ X_{n-1}.$$

Consider the part X_rX_{r+1}.

Its distance from O lies between OX_s and OX_{s+1}, *i.e.* between

$$s\frac{l}{n} \text{ and } (s+1)\frac{l}{n},$$

and therefore the "mr^2" for this part lies between

$$\frac{M}{n}\left(s\frac{l}{n}\right)^2 \text{ and } \frac{M}{n}\left((s+1)\frac{l}{n}\right)^2.$$

The "mr^2" for the first part lies between

$$0 \text{ and } \frac{Ml^2}{n^3}\,.\,1^2.$$

The "mr^2" for the second part lies between

$$\frac{Ml^2}{n^3}\,.\,1^2 \text{ and } \frac{Ml^2}{n^3}\,.\,2^2,$$

etc.

The "mr^2" for the last part lies between

$$\frac{Ml^2}{n^3}(n-1)^2 \text{ and } \frac{Ml^2}{n^3}\,.\,n^2.$$

Hence by addition,

Σmr^2 lies between $\dfrac{Ml^2}{n^3}\{0^2+1^2+2^2+\ldots\ldots(n-1)^2\}$

and $\dfrac{Ml^2}{n^3}\{1^2+2^2+3^2+\ldots\ldots n^2\}$,

i.e. between $\dfrac{Ml^2}{n^3}\,.\,\dfrac{(n-1)\,n\,(2n-1)}{6}$

and $\dfrac{Ml^2}{n^3}\,.\,\dfrac{n\,(n+1)\,(2n+1)}{6}$,

i.e. between $\dfrac{Ml^2}{6}\left(2-\dfrac{3}{n}+\dfrac{1}{n^2}\right)$ and $\dfrac{Ml^2}{6}\left(2+\dfrac{3}{n}+\dfrac{1}{n^2}\right)$.

Now let us suppose that n becomes very large indeed: then $\dfrac{3}{n}$, $\dfrac{1}{n^2}$ both become very small indeed and may be neglected. The last two expressions will then each be equal to $\dfrac{Ml^2}{6}\,.\,2$, *i.e.* $\dfrac{Ml^2}{3}$.

Hence $\Sigma mr^2 = \dfrac{Ml^2}{3}$ in the case of a uniform rod.

75. Definition of Moment of Inertia.—Consider a body A of mass M, and any line XY. Regard the body A as made up of a very large number of small portions of masses $m_1, m_2 \ldots$ Call the distance of these portions from the line XY $r_1, r_2, r_3 \ldots$. Then the sum of the products

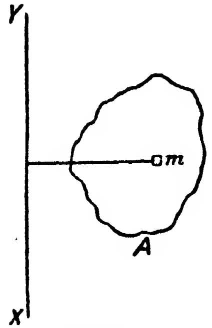


Fig. 35.

$$m_1 r_1^2 + m_2 r_2^2 + \ldots$$

is called the moment of inertia of the body A about the line XY. Denote it by I.

Then $I = m_1 r_1^2 + m_2 r_2^2 + \ldots$

or $I = \Sigma m r^2$.

Now suppose $\Sigma m r^2 = Mk^2$,

where $M = m_1 + m_2 + \ldots = \Sigma m$, then k is termed the radius of gyration.

$$I = \Sigma m r^2 = Mk^2.$$

76. Particular Values of k^2.—In Art. 74 we found the value of the moment of inertia of a rod about a line perpendicular to the rod passing through the end of the rod.

We often speak of the moment of inertia of a plane figure about a point: by this is really meant the moment of inertia about a line drawn through the point perpendicular to the plane of the figure. Some particular cases of moments of inertia should be known.

(1) Moment of inertia of a rod of length l about one end $= M\dfrac{l^2}{3}$. This was proved in Art. 74.

(2) Moment of inertia of a rod about its centre $= M\dfrac{l^2}{12}$.

(3) Moment of inertia of a rectangular bar about an axis through its centre of mass parallel to one of the edges

$$= M\frac{\text{Sum of squares on other edges}}{12}$$

$$= M\frac{b^2 + c^2}{12} \text{ (Fig. 36).}$$

(2) is a particular case of (3).

(4) Moment of inertia of a hoop about its centre

$$= Mr^2.$$

(5) Moment of inertia of a disc about its centre

$$= Mr^2/2.$$

(6) Moment of inertia of a sphere about its diameter

$$= M \,.\, 2r^2/5.$$

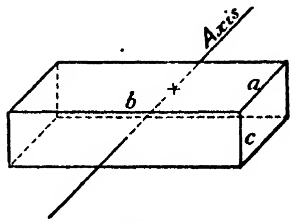


Fig. 36.

All these results may be found by the Integral Calculus from the formula $I = \int \rho r^2 dv$, where dv is an element of the volume of the body of density ρ, distant r from the axis. (See Arts. 76a–76c.)

76a. Principle of Perpendicular Axes.—If the moments of inertia of a lamina about two axes in its own plane at right angles to each other are I_1 and I_2, the moment of inertia of the lamina, I, about an axis passing through the point of intersection of the above axes and perpendicular to the plane of the lamina is given by $I = I_1 + I_2$.

Proof. — Let OX, OY (Fig. 36*a*) be the two axes at right angles to each other. Let m_1 be an element of mass at distances x_1, y_1, from the axes OY, OX, and at distance p_1 from the axis through O perpendicular to plane XOY.

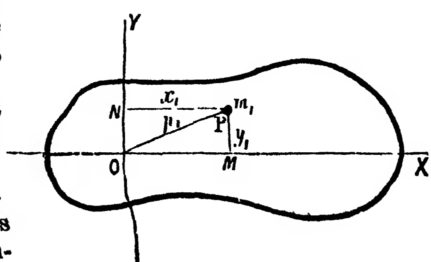


Fig. 36*a*.

$$\begin{aligned} \text{Then } I = \Sigma m_1 p_1^2 &= \Sigma m_1 (x_1^2 + y_1^2) \\ &= \Sigma m_1 x_1^2 + \Sigma m_1 y_1^2 \\ &= I_1 + I_2. \end{aligned}$$

76b. The Principle of Parallel Axes.—The moment of inertia of a lamina about any axis in its own plane is equal to the moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the lamina into the square of the distance between the two axes.

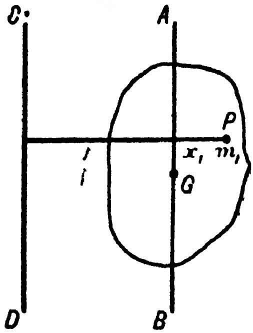


Fig. 36*b*.

Proof.—Let CD (Fig. 36*b*) be the axis in the plane of the lamina and AB the parallel axis through G, the centre of gravity mass of the lamina. Let M be the mass of the lamina and m_1 the mass of an element of the body at a point P, whose distance from AB is x_1. The signs of the distances of such elements from AB are reckoned positive or negative according as the particles are on the positive or negative side of AB. Moment of inertia of m_1 about CD

$$\begin{aligned} &= m_1(x_1 + h)^2 \\ &= m_1 (x_1^2 + h^2 + 2x_1 h) \\ &= m_1 x_1^2 + m_1 h^2 + 2\, h m_1 x_1. \end{aligned}$$

Let I and I_0 be the moments of inertia about CD and AB respectively, then

$$\begin{aligned} I &= \Sigma m_1 (x_1 + h)^2 \\ &= \Sigma m_1 x_1^2 + \Sigma m_1 h^2 + 2\, \Sigma m_1 x_1 h \\ &= I_0 + Mh^2 + 2h\Sigma m_1 x_1. \end{aligned}$$

Now, since AB passes through G, the centre of mass, the lamina will balance on a knife edge immediately under AB.

$$\therefore\ \Sigma m_1 x_1 = 0.$$
$$\therefore\ I = I_0 + Mh^2.$$

This theorem is also true for the moment of inertia of a solid body about any axis, or of a lamina about an axis not in its own plane. The proof is a slight extension of the above.

76c. Important Cases of Moments of Inertia.—(1) *Moment of inertia of a rectangle of sides a, b, about an axis through its centre parallel to the side a.*

Let M be the mass of the rectangle. Divide the rectangle into strips parallel to this axis. Consider a strip of breadth dx at a distance x from the axis. The mass of this strip is

$$\frac{dx}{b}M.$$

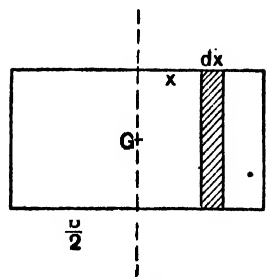


Fig. 36c.

Its moment of inertia is

$$\frac{dx}{b}M \cdot x^2.$$

Thus moment of inertia of the rectangle

$$= 2\int_0^{\frac{b}{2}} \frac{dx}{b}Mx^2 = 2\frac{M}{b}\left[\frac{x^3}{3}\right]_0^{\frac{b}{2}} = M\frac{b^2}{12}.$$

Corollary.—If a is small, making $b = l$, we see that the moment of inertia of rod of length l about axis through its middle perpendicular to its length is $M\frac{l^2}{12}$ (see § 76).

(2) *Moment of inertia of same rectangle about side a.*

Use the Principle of Parallel Axes.

Moment of inertia about the parallel axes through G is $Mb^2/12$. If the mass is collected at G its moment of inertia about the side a is $Mb^2/4$.

Thus required moment of inertia

$$= M\frac{b^2}{12} + M\frac{b^2}{4} = M\frac{b^2}{3}. \qquad \text{[Compare § 76 (1).}$$

(3) *Moment of inertia of a rectangle of sides a, b, about an axis through its centre perpendicular to the plane of the rectangle.*

Apply the *principle of perpendicular axes.* Then the moment of inertia about the given axis = the moment of inertia about an axis through the centre parallel to the sides of length a, plus the moment of inertia about

an axis through the centre parallel to the sides of length b, and hence

$$= \frac{Mb^2}{12} + \frac{Ma^2}{12} = M\frac{a^2 + b^2}{12}.$$

(4) *Moment of inertia of a disc about its centre.* Let M be mass of the disc and r its radius. Consider an elementary ring of radius x (Fig. 36d) and width dx. Its area is equal to the product of its circumference and its width, *i.e.* to $2\pi x \,.\, dx$. Its mass

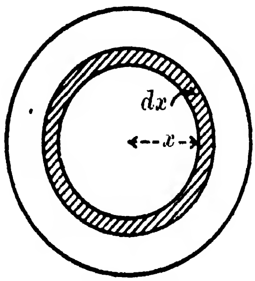


Fig. 36d.

$$= \frac{2\pi x \,.\, dx}{\pi r^2} M,$$

and its moment of inertia about an axis through its centre perpendicular to its plane

$$= \frac{2\pi x dx \,.}{\pi r^2} M \,.\, x^2.$$

$\therefore$ $M.I.$ of whole disc about this axis

$$= \int_0^r \frac{2\pi x dx}{\pi r^2} Mx^2 = \frac{2}{r^2} M \int_0^r x^3 dx$$

$$= \frac{2}{r^2} M \left[\frac{x^4}{4}\right]_0^r = \frac{Mr^2}{2}.$$

(5) *Moment of inertia of a disc about a diameter.* If I is the required moment of inertia, I is also the moment of inertia about a perpendicular diameter, and by *the principle of perpendicular axes*

$$I + I = \frac{Mr^2}{2}, \quad \therefore I = \frac{Mr^2}{4}.$$

(6) *Moment of inertia of a sphere about a diameter.* Let M be the mass and r the radius. The mass of unit volume is therefore

$$M \Big/ \frac{4}{3}\pi r^3 = 3M \Big/ 4\pi r^3.$$

Take a thin circular slice at a distance x (Fig. 36e) from the centre, and of thickness dx. Its radius $=\sqrt{r^2-x^2}$, its volume $=\pi(r^2-x^2)dx$

$\therefore$ its mass $=\pi(r^2-x^2)\,.\,dx\,.\,\frac{3M}{4\pi r^3}$.

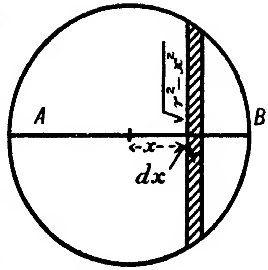


Fig. 36e.

Its moment of inertia about diameter $AB=\frac{\pi(r^2-x^2)dx}{4\pi r^3}\,.\,3M\,.\,\frac{r^2-x^2}{2}$ by (4) above.

$\therefore$ Moment of inertia of sphere about AB

$$=2\times\frac{3\pi M}{8\pi r^3}\int_{x=0}^{x=r}(r^2-x^2)^2dx$$

$$=\frac{3M}{4r^3}\int_{x=0}^{x=r}(r^4-2r^2x^2+x^4)dx.$$

$$=\frac{3M}{4r^3}\left[r^5-\frac{2r^5}{3}+\frac{r^5}{5}\right]$$

$$=\frac{3M}{4r^3}\cdot\frac{8r^5}{15}=\frac{2}{5}Mr^2.$$

77. Energy of Rotation of a Body.—Suppose a body is rotating about any line passing through its centre of mass. It has kinetic energy due to its motion: this energy we shall call energy of rotation. To calculate this energy imagine that the body is divided up into a large number of small pieces of masses m_1, m_2, . . . at distance r_1, r_2, . . . from the axis of rotation which passes through the centre of mass. If ω is the angular velocity, the linear velocity of m_1 is $r_1\omega$ and its kinetic energy is therefore $\frac{1}{2}m_1r_1^2\omega^2$. Hence the total energy of rotation

$$=\tfrac{1}{2}m_1r_1^2\omega^2+\tfrac{1}{2}m_2r_2^2\omega^2+\ldots$$
$$=\tfrac{1}{2}\omega^2\{m_1r_1^2+m_2r_2^2+\ldots\}$$
$$=\tfrac{1}{2}\omega^2\Sigma mr^2$$
$$=\tfrac{1}{2}\omega^2\,I\text{ or }\tfrac{1}{2}Mk^2\omega^2.$$

78. Total Kinetic Energy of a System.—Suppose a body is not only rotating, but that the centre of mass has also a velocity. As an example of this take the case of a penny rolling down an inclined plane.

The energy of such a body may be shown to be equal to the sum of two quantities, (1) the energy of translation, as we shall call it, and (2) the energy of rotation.

(1) The energy of translation is the same as the energy of a body of equal mass moving with the velocity of the centre of mass.

(2) The energy of rotation is the energy that the body would have were the centre of mass at rest and the parts of the body revolving round it with the same *angular* velocity as before.

Total kinetic energy of a body

= energy of translation + energy of rotation.

The total kinetic energy of a system of bodies (*e.g.* Jupiter and his satellites) is equal to the sum of the kinetic energy of a body of mass equal to that of the whole system moving with the velocity of the centre of mass, and of the kinetic energy of the separate parts due to motion relative to the centre of mass.

79. Energy of a Penny rolling on a Table.—An example will make this clear. Suppose a penny mass M, radius r, roll on a table, and that its centre of mass moves with a velocity v. In going a distance $2\pi r$ the penny turns round once—*i.e.* it describes an angle 2π about its centre The time it takes to describe this angle is

$$\frac{2\pi r}{v},$$

so that the angular velocity of a penny about its centre is

$$\frac{2\pi}{\frac{2\pi r}{v}} = \frac{v}{r}.$$

If we call this ω, then $v = r\omega$.

The energy of translation $= \frac{1}{2} Mv^2$.

The energy of rotation $= \frac{1}{2} Mk^2\omega^2$ (Art. 77.)

$= \frac{1}{2} M \frac{r^2}{2} \omega^2.$ (Art. 76.)

$= \frac{1}{4} Mv^2.$

Hence the total energy $= \frac{1}{2} Mv^2 + \frac{1}{4} Mv^2$

$= \frac{3}{4} Mv^2$ or $\frac{3}{4} Mr^2\omega^2.$

80. **Motion of a Disc on an Inclined Plane.**—We suppose the plane rough so that the penny rolls without slipping. Use the notation indicated in Fig. 37. Suppose the disc starting from rest has moved down the plane a distance s, in a time t, and that its velocity is now v. Since the body has rolled down a distance s, it has descended a distance $s \sin \alpha$. The work done by gravity is therefore $Mgs \sin \alpha$. This must be equal to the gain of kinetic energy of the body (no work is done by friction since there is no slipping), *i.e.* to $\frac{3}{4} Mv^2$;

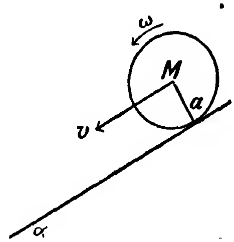


Fig. 37.

$$\therefore \tfrac{3}{4} Mv^2 = Mgs \sin \alpha$$

$$\therefore v^2 = \tfrac{4}{3} gs \sin \alpha$$

i.e. $$v^2 = 2(\tfrac{2}{3} g \sin \alpha)\, s.$$

Compare this with the kinematical formula $v^2 = 2fs$, and we see that the motion of the disc is similar to that of a body moving with a constant acceleration $f = \frac{2}{3} g \sin \alpha$.

81. **General Case of a Body rolling down an Inclined Plane.**—Adopting the same notation as in the preceding article, and equating the gain of kinetic energy to work done by gravity, we get

$$Mgs \sin \alpha = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} Mk^2\omega^2$$

$$= \tfrac{1}{2} Mv^2 \left(1 + \frac{k^2}{a^2}\right)$$

or $$v^2 = 2\left(\frac{a^2}{a^2 + k^2} g \sin \alpha\right) s.$$

Hence the acceleration down the plane $= \dfrac{a^2}{a^2 + k^2} g \sin a$.

From this relation it is obvious that the greater k is compared with a, the less the acceleration of the body: hence a ball will roll faster than a disc and a disc faster than a hoop.

This result applies to any symmetrical body rolling down a plane when the centre of mass describes a straight line.

EXAMPLES VI.

1. Investigate the motion of a wheel and axle.

Take the axle of radius r, length l, density ρ; the wheel of radius R, thickness b, density σ, and mass W. Let the mass attached to the rope on the axle be M, that on the rope round the wheel m.

Suppose that after a certain time the mass m has descended a distance s: then M has ascended a distance $\dfrac{r}{R}s$, so that the total

work done by gravity $= mgs - M\dfrac{r}{R}gs$.

Call ω the angular velocity of the wheel and axle.

The energy of the axle $= \frac{1}{2}\dfrac{r^2}{2} \cdot \omega^2 l\pi r^2\rho$.

The wheel is cut out at the centre for the axle to pass through, so

that its moment of inertia $= \dfrac{R^2}{2} \cdot \pi R^2 b\sigma - \dfrac{r^2}{2}\pi r^2 b\sigma$

$$= \frac{\pi b\sigma}{2}(R^4 - r^4),$$

so that its energy of rotation

$$= \tfrac{1}{2}\frac{\pi b\sigma}{2}(R^4 - r^4)\omega^2 = \tfrac{1}{4}W(R^2 + r^2)\omega^2$$

Equate the work done to the energy acquired,

$$gs\left(m - \frac{rM}{R}\right) = \frac{\pi}{4}\omega^2\left[b\sigma(R^4 - r^4) + lr^4\rho\right].$$

Hence if v is the velocity of the mass m and f its acceleration,

$$\text{then} \quad v = R\omega \text{ and } f = \frac{v^2}{2s} = \frac{2\left(m - \frac{rM}{R}\right)R^2 g}{\pi[b\sigma(R^4 - r^4) + lr^4\rho}$$

$$= \frac{2\left(m - \frac{rM}{R}\right)R^2 g}{W(R^2 + r^2) + \pi r^4 l\rho}.$$

2. A hollow spherical shell is placed on an inclined plane. Find how long it will take to travel a distance s.

If R, r are the external and internal radii of the shell, its moment of inertia

$$= \frac{2}{5}M\frac{R^5 - r^5}{R^3 - r^3}.$$

Its acceleration

$$= f = \frac{R^2}{R^2 + \frac{2}{5}\frac{(R^5 - r^5)}{R^3 - r^3}} g \sin a.$$

The time taken to travel a distance s is given by the relation

$$s = \tfrac{1}{2}ft^2,$$

$$\therefore t = \sqrt{\frac{2s}{f}},$$

where f has the value given above.

3. A rod 3 ft. long, weight 10 lbs., revolves 50 times a minute about one end. Find its kinetic energy.

4. A conical pendulum is 10 ft. long; the semi-vertical angle of the cone is 30°; the mass of the bob is 12 lbs. Find the tension of the thread and the period.

5. A flywheel of mass $2\frac{1}{2}$ tons, 8 ft. diameter, makes 250 revolutions per minute. Find (1) its angular velocity, (2) its energy, (3) its moment of inertia. Assume the mass is concentrated at the rim.

6. Explain how a conical pendulum is used to regulate the speed of a machine driven by a steam engine.

7. What is meant by the kinetic energy of a body? How is it calculated (*a*) when the body moves along a straight line, (*b*) when the body rotates round a fixed axis?

A thin hoop of mass 5 kilogrammes and radius 50 cm. rolls along the ground at the rate of 10 metres per second. Calculate (in ergs) its kinetic energy.

8. State Newton's second law of motion. What is the analogous law for rotation round an axis?

9. Will a garden roller run down hill faster or slower if the weight inside it on the bottom of the handle is replaced by an equal weight attached to the iron cylinder?

Give reasons for your answer.

10. Two spheres of the same mass are externally exactly similar, but one is hollow while the other is solid. Explain clearly how they could be distinguished.

11. A ball of mass $\frac{1}{2}$ lb. is fastened to one end of a piece of elastic, of which the unstretched length is 1 ft. To stretch the elastic a distance a requires a pull of λa lbs. wt. Find what the length of the string will be when the ball revolves at the end of the elastic at 200 revolutions per minute.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER VII.

THE PENDULUM AND SIMPLE HARMONIC MOTION.

82. Isochronous Motion.—If a tuning-fork or fiddle-string is plucked it gives out a note the pitch of which remains constant though the loudness of the sound gradually diminishes. This indicates that the time in which vibrations from side to side are executed remains unaltered, while the extent of the motion decreases. Such a motion is termed isochronous from the fact that its period is constant, and harmonic from its connection with musical instruments.

83. Projection of Circular Motion.—Let a particle P, Fig. 38, describe a circle, centre O, with uniform angular velocity. Take any diameter AA' of this circle, and from P draw PN perpendicular to AA'. The point N is called the projection of P on AA'.

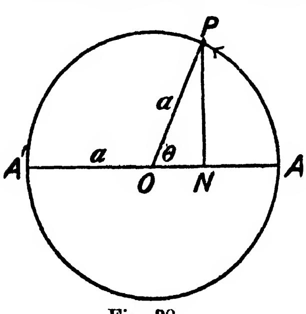


Fig. 38.

Now as P moves round and round the circumference of the circle, the point N will move up and down the diameter AA'. We are concerned with the motion of the point N.

The length AO is termed the *amplitude*.

Let a = radius of circle,

ω = angular velocity of P about the centre O,

θ = angle AOP.

The acceleration of P is along PO and is equal to $a\omega^2$.

The resolved part of this parallel to AA'

$$= a\omega^2 \cos AOP$$

$$= a\omega^2 \frac{ON}{OP}$$

$$= \omega^2 . ON.$$

Now the acceleration of N towards O is the same as the resolved part of the acceleration of P in the direction AA'. Hence the acceleration of N towards the point O is equal to $\omega^2 . ON$.

The velocity of the point P is equal to $a\omega$, and is directed along the tangent at P.

Hence the velocity of N = resolved part of this along NO

$$= a\omega \sin \theta$$

$$= \omega . PN.$$

The period of N — *i.e.* the time N takes to pass from A to A' and back—is the time which P takes to pass right round the circle.

A body moving with angular velocity ω describes an angle 2π in time $\frac{2\pi}{\omega}$,

$$\therefore \text{ period of } N = \frac{2\pi}{\omega}.$$

The oscillating motion of the point N is termed *Simple Harmonic Motion.*

We note particularly that—

(1) The acceleration ($\omega^2 . ON$) is always directed towards the centre, and is proportional to the distance from the centre.

(2) The period $\left(\frac{2\pi}{\omega}\right)$ is independent of the amplitude: it depends only on the angular velocity of the point P about the centre.

84. Simple Harmonic Motion.—Let a body N be constrained to move along any path AOA' (Fig. 39), and let its motion be such that its acceleration along its path is always proportional to its distance (measured along the path) from some fixed point O in its path.

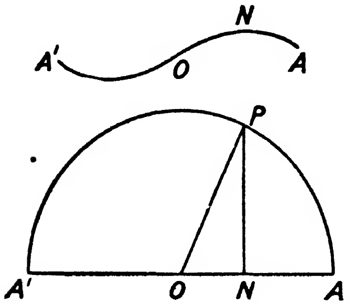


Fig. 39.

Then the point N has simple harmonic motion.

Under the conditions stated, the motion of N along its path is independent of the nature of the path, whether straight or curved. If the path is curved imagine it straightened out as in the lower part of Fig. 39.

The acceleration of N towards O is proportional to ON: suppose it is equal to λON where λ is some constant.

Suppose A is the extreme end of the path.

With O as centre and OA as radius describe a circle APA'.

From the position in which N is at any instant draw NP perpendicular to OA, cutting the circle in P.

Now if P move round the circle with uniform angular velocity $\sqrt{\lambda}$, then the acceleration of P resolved in the direction AO is by Art. 83 equal to $\lambda \,.\, ON$, and the particle N will move backwards and forwards along AOA' as P describes the circle.

Now the period of P is $\dfrac{2\pi}{\sqrt{\lambda}}$; this must be the period of N. Hence if a body move in a path with its acceleration always directed to a fixed point O in its path and equal to λ times its distance from that point, its motion is simple harmonic motion, and of period $\dfrac{2\pi}{\sqrt{\lambda}}$.

Simple harmonic motion is therefore isochronous: the period is independent of the amplitude of the motion.

85. The motion may also be investigated from definition as follows:

Let s be the distance of N from O at a time t.

$$\text{Then velocity of } N \text{ along } ON = \frac{ds}{dt}.$$

$$\text{Acceleration of } N \text{ along its path } towards\ O = -\frac{d^2s}{dt^2};$$

this by hypothesis is proportional to the distance ON, call it $\lambda \, . \, ON$.

$$\therefore \frac{d^2s}{dt^2} = -\lambda s.$$

The complete solution of this differential equation is

$$s = A \cos \sqrt{\lambda}t + B \sin \sqrt{\lambda}t,$$

where the constants A, B are to be determined from the conditions of the motion. If, for instance, the body starts from O, then $s = o$ when $t = o$, and we have $O = A \cos o + B \sin o$, i.e. $A = O$, so that the equation becomes $s = B \sin \sqrt{\lambda}t$.

If in addition we know that the initial velocity is V, then

$$V = \left(\frac{ds}{dt}\right)_{t=o} = B\left(\sqrt{\lambda} \cos \sqrt{\lambda}t\right)_{t=o} = B\sqrt{\lambda},$$

i.e.
$$B = \frac{V}{\sqrt{\lambda}},$$

and the equation finally becomes $s = \dfrac{V}{\sqrt{\lambda}} \sin \sqrt{\lambda}t$.

From the general relation $s = A \cos \sqrt{\lambda}t + B \sin \sqrt{\lambda}t$

we deduce $\quad v = \dfrac{ds}{dt} = -A\sqrt{\lambda} \sin \sqrt{\lambda}t + B\sqrt{\lambda} \cos \sqrt{\lambda}t.$

Hence we find that the period is $\dfrac{2\pi}{\sqrt{\lambda}}$, for if we increase t by that amount we get

$$s = A \cos \sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right) + B \sin \sqrt{\lambda}\left(t + \frac{2\pi}{\sqrt{\lambda}}\right)$$
$$= A \cos \sqrt{\lambda}t + B \sin \sqrt{\lambda}t.$$

and v also repeats its value.

86. The Simple Pendulum.—The motion of the pendulum is nearly isochronous and simple harmonic.

The simple pendulum is a small heavy particle at one end of a weightless string, the other end of which is fixed. The particle or bob swings in a vertical plane under the action of gravity. Of course a simple pendulum is merely ideal. Every actual pendulum is really a compound pendulum, *i.e.* its mass is not a point mass. Still the motion of the centre of a spherical metal bob attached to a piece

of cotton is almost that of the ideal simple pendulum. Suppose Fig. 40 represents such a pendulum. In this OB is the string, B the bob.

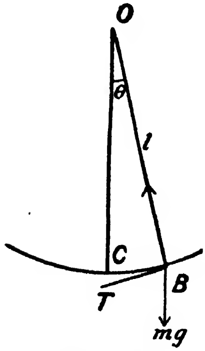


Fig. 40.

If the mass of the bob is m and θ is the inclination of OB to the vertical, then the forces acting on the bob are (1) its weight mg vertically downwards, (2) the tension of the string.

Now B describes a circular arc: its acceleration is not entirely directed along the tangent, BT, but its component in the direction BT multiplied by the mass m must be equal to the component of the resultant force in that direction.

The component of mg along BT is

$$mg \sin \theta.$$

The pull of the string is perpendicular to BT and its component along BT is zero. Hence acceleration of bob along tangent to path

$$= g \sin \theta$$

$$= g\left(\theta - \frac{\theta^3}{\underline{|3}} + \dots\right)$$

$$= g\,\theta, \text{ if } \theta \text{ is small}$$

$$= g\,\frac{\text{arc } BC}{l}$$

$$= \frac{g}{l} \text{ arc } BC.$$

Hence the acceleration of B along its path is equal to $\frac{g}{l}$ times its distance from a fixed point C in its path.

The motion is therefore simple harmonic and the period is

$$\frac{2\pi}{\sqrt{\frac{g}{l}}}, \text{ i.e. } 2\pi\sqrt{\frac{l}{g}}.$$

87. In this investigation we have neglected θ^3 and higher powers. The motion of a simple pendulum is therefore only isochronous if the angle which the string traces out is small. If, then, this angle is not small the result $t = 2\pi\sqrt{\dfrac{l}{g}}$ is no longer true. A nearer approximation is given by the relation $t = 2\pi\sqrt{\dfrac{l}{g}}\left(1 + \dfrac{a^2}{16}\right)$, where a is the half-angle of swing. The solution of the problem in this case is hardly within our range, but results showing the variation of period (t) with the half-angle of swing (a) may be of some interest.

a	t
0	1
2	1·00008
4	1·0005
10	1·002
20	1·008
30	1·02
60	1·07
90	1·18

88. Galileo.—The first to make use of the pendulum appears to have been Galileo. In the cathedral at Pisa, 1583, he watched a swinging lamp and noticed that as the oscillations of the lamp died away their period still remained the same. He timed the period by beats of his pulse. He pointed out that this discovery could be used to govern clocks.

89. Cycloidal Pendulum.—The motion of the ordinary pendulum is not quite isochronous. That of the cycloidal pendulum is. The cycloid is the curve traced by a point on the circumference of a wheel rolling on a plane surface; *e.g.* by a speck of mud on the rim of a cart wheel. Let $CBADE$ (Fig. 41) be such a curve. As represented in the figure the generating circle rolls on the under side of the line XY and comes into contact with XY at A.

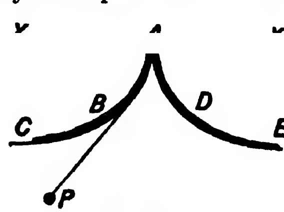


Fig. 41.

Suppose cheeks of metal are cut to this shape and a pendulum bob P is supported by the string ABP which wraps round on these cheeks as it swings from side to side. Then if the length of ABP is half that of the cycloid, it can be shown that P itself traces out another cycloid, and its motion is strictly isochronous and simple harmonic.

90. It may be useful to investigate the motion of the simple pendulum from first principles.

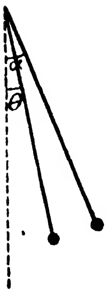


Fig. 42.

Suppose the bob is started by being held initially with the string inclined at an angle α to the vertical and released. (Fig. 42.)

After a time t suppose it makes an angle θ with the vertical.

The distance the bob has descended is $l(\cos\theta - \cos\alpha)$.

The angular velocity of the bob is $\frac{d\theta}{dt} = \omega$, so that the kinetic energy of the bob is $\frac{1}{2}ml^2\omega^2$.

Equate this to the work done by gravity and we get

$$mgl(\cos\theta - \cos\alpha) = \tfrac{1}{2}ml^2\omega^2$$

$$g(\cos\theta - \cos\alpha) = \tfrac{1}{2}l\omega^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Differentiate by t,

$$\therefore \quad -g\sin\theta \,.\, \frac{d\theta}{dt} = l\omega\,\frac{d\omega}{dt}\,;$$

$$\text{but} \qquad \omega = \frac{d\theta}{dt},$$

$$\therefore \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

$$\text{Hence} \qquad g\sin\theta + l\frac{d^2\theta}{dt^2} = 0.$$

$$\text{Hence, if } \theta \text{ is small,} \qquad \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0,$$

i.e. the acceleration is proportional to the arc, so that the motion is harmonic and the period is

$$2\pi\sqrt{\frac{l}{g}}.$$

The period of a pendulum as given in the above relation is the time to *and* fro. In clocks what is spoken of as the time of the pendulum is the time to *or* fro. The period of what is known as a seconds pendulum is on our notation two seconds.

91. The Compound Pendulum.—Suppose a body is capable of free rotation round a horizontal axis. Let G be its centre of gravity.

Suppose Fig. 43 represents a vertical section of the body taken through the centre of gravity at right angles to the axis. Let the axis cut the plane of the paper at O. In the position of equilibrium G will be vertically below O. Suppose the body was initially displaced so that the

line OG made an angle α with the vertical. After the lapse of a time t, let the angle that OG makes with the vertical be denoted by θ.

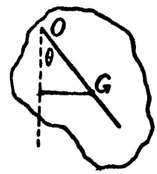


Fig. 43.

Call the angular velocity of the body at this instant ω.

Then $$\omega = \frac{d\theta}{dt}.$$

Let $OG = l$.

The depth of G below O is $l \cos \theta$, *i.e.* the centre of gravity is lower by $l(\cos \theta - \cos \alpha)$ than it was in its initial position. Hence the work done on it by gravity is
$$Mgl(\cos \theta - \cos \alpha),$$
where M is the mass of the body.

Imagine the body composed of small particles of masses $m_1, m_2, m_3, \ldots$ situated at distance $r_1, r_2, r_3, \ldots$ from the axis through O. Each of these is at the instant considered revolving round the axis with an angular velocity ω. The linear velocities of the particles are therefore $r_1\omega, r_2\omega, r_3\omega, \ldots$ and their kinetic energies are
$$\tfrac{1}{2}m_1r_1^2\omega^2, \quad \tfrac{1}{2}m_2r_2^2\omega^2, \quad \tfrac{1}{2}m_3r_3^2\omega^2 \ldots$$

But the total kinetic energy must be equal to the work done on the body by gravity: hence
$$\begin{aligned} Mgl(\cos \theta - \cos \alpha) &= \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \ldots \\ &= \tfrac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + \ldots) \\ &= \tfrac{1}{2}\omega^2 Mk^2 \end{aligned}$$
where Mk^2 is the moment of inertia, k is the radius of gyration.

Divide by Ml and we get
$$g(\cos \theta - \cos \alpha) = \tfrac{1}{2}\left(\frac{k^2}{l}\right)\omega^2 \qquad \text{.......................(1)}$$

This equation is of exactly the same form as that of (1) in Art. 90. Hence the motion of the compound pendulum is exactly the same as that of a simple pendulum of length $\frac{k^2}{l}$.

Its period is therefore given by the relation
$$t = 2\pi\sqrt{\frac{k^2}{lg}} \qquad \text{..........................(2)}$$

92. Expression for t in terms of moment of inertia round the centre of gravity.—The energy of the pendulum in Art. 91 was found to be $\frac{1}{2}\omega^2 Mk^2$, where Mk^2 was the moment of inertia round the axis through O. We can express this in a different form.

The linear velocity of the centre of gravity is $l\omega$, so that the energy of translation is $\frac{1}{2}Ml^2\omega^2$.

The body is rotating relatively to G with an angular velocity ω, so that the energy of rotation about G is $\frac{1}{2}MK^2\omega^2$, where MK^2 is the moment of inertia about a line through G parallel to the axis of suspension through O (Art. 78). Hence

$$\tfrac{1}{2}Mk^2\omega^2 = \tfrac{1}{2}Ml^2\omega^2 + \tfrac{1}{2}MK^2\omega^2,$$

i.e. $$k^2 = l^2 + K^2.$$

The relation (2) of Art. 91 then becomes

$$t = 2\pi\sqrt{\frac{l^2 + K^2}{lg}} \quad \text{..........................(3)}$$

when l is the distance of the centre of gravity from the axis of revolution, and K is the radius of gyration of the pendulum about a parallel axis through the centre of gravity.

We might have obtained the relation (3) more quickly had we assumed the proposition that the moment of inertia of a body about any axis is equal to the moment of inertia of the body about a parallel axis through the centre of mass, together with the moment of inertia about one of these axes of a mass equal to that of the body and situated on the other axis.

With our notation $Mk^2 = MK^2 + Ml^2$.

93. Centres of Oscillation and Suspension.—The point O in which the axis meets the vertical plane through the centre of gravity is called the *centre of suspension.*

Take a point C in OG such that the length OC is equal to that of a simple pendulum that would vibrate in the same time as the compound pendulum. Call this length L,

then $$2\pi\sqrt{\frac{L}{g}} = t = 2\pi\sqrt{\frac{l^2 + K^2}{lg}}$$

$$\therefore \quad L = \frac{l^2 + K^2}{l} \quad \text{..........................(4)}$$

C is called the *centre of oscillation.*

Now suppose the pendulum were inverted and allowed to swing about an axis through C parallel to the original axis through O.

$$CG = L - l.$$

∴ The time of swing about C

$$= 2\pi\sqrt{\frac{(L-l)^2 + K^2}{(L-l)g}}$$

$$= 2\pi\sqrt{\frac{\left(\frac{K^2}{l}\right)^2 + K^2}{\frac{K^2}{l}g}}.$$

since by (4) $L - l = \dfrac{K^2}{l}$

$$= 2\pi\sqrt{\frac{l^2 + K^2}{lg}}, \quad\text{.................. (5)}$$

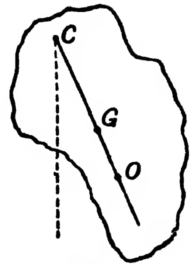


Fig. 44.

i.e. the time of swing is the same as when the pendulum was oscillating on the axis O. Hence the centres of oscillation and suspension are interchangeable.

94. Kater's Pendulum.—Let us suppose a pendulum made as in Fig. 45, which is capable of oscillating about either of the two parallel knife-edges A or B, and that these two knife-edges are on opposite sides of the centre of mass G.

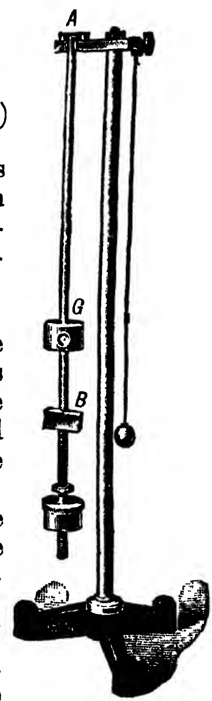


Fig. 45.

Then if the time of swing of the pendulum about A is the same as the time about B, and A, B are not equidistant from G, then the points A and B must correspond with the points C, O of Art. 93, and the distance AB must as in Art. 91 be equal to the length of the simple equivalent pendulum. (See Example 1, p. 87.)

This system was adopted by Captain Kater to determine

the length of the seconds pendulum. In order to arrange the axes A and B in such a way that the times of vibration about them might be equal, the pendulum was built with a sliding ring regulated by a screw. By moving this sliding ring up or down the position of the centre of mass could be altered until the time became exactly equal.

Kater thought his determination of the length of the seconds pendulum would be useful as a standard of measurement, so that he determined it with great accuracy. By a fire in the Houses of Parliament (1832) the standard yard was partially destroyed, but a commission appointed to recover the length decided that pendulum methods afforded measurements of less accuracy than those that could be obtained from copies that had previously been taken from the standard yard. From these copies a new standard was made, which still remains the legal unit of length.

95. Oscillation of a Body on an Elastic String.—Suppose a body of mass M is attached to a vertical elastic string. If the body is displaced vertically it will oscillate, and will execute simple harmonic vibrations provided that Hooke's Law (Art. 108) is obeyed, *i.e.* provided that the tension of the string is proportional to its extension.

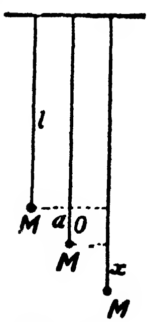


Fig. 46.

Suppose the natural length of the string is l, and that it is stretched a distance a when it supports the mass M. The tension of the string will be $\lambda \frac{a}{l}$, where λ is the modulus of the string.

$$\text{Hence } \lambda\frac{a}{l} = Mg.$$

Now suppose the mass M oscillates, and that at any time the string is stretched a further distance x below the equilibrium position O. The tension of the string is

$$\lambda\frac{a+x}{l}.$$

The only other force acting on M is the weight.

The resultant force $= \lambda \dfrac{a+x}{l} - Mg$

$$= \lambda \frac{a+x}{l} - \lambda \frac{a}{l}$$

$$= \lambda \frac{x}{l},$$

and is directed towards O.

Hence the acceleration of M towards $O = \dfrac{\lambda}{lM} x$.

Hence the acceleration of M is always directed towards O, and is proportional to the distance from O.

The motion is therefore simple harmonic and the period is

$$2\pi\sqrt{\frac{lM}{\lambda}} \quad \text{..........................(6)}$$

$$\text{Since } Mg = \lambda \frac{a}{l},$$

$$\therefore \quad \frac{lM}{\lambda} = \frac{a}{g},$$

$$\therefore \quad \text{period} = 2\pi\sqrt{\frac{a}{g}} \quad \text{....................(7)}$$

i.e. the period is that of a simple pendulum whose length is a, the amount by which M stretches the spring.

To find λ we may find, as in Art. 107, the extension due to an increase of load by a mass m. Call this b.

Then $$mg = \lambda \frac{b}{l}.$$

Substitute in the expression obtained for the period and we get the relation

$$t = 2\pi\sqrt{\frac{bM}{gm}}.$$

In this relation the mass of the string has been supposed small compared with M. If in experimental work higher accuracy is required we must add to the mass M a third of

the mass of the spring. Call this S and the relation becomes

$$t = 2\pi\sqrt{\frac{b}{mg}\left(M + \frac{S}{3}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

This relation may be used for finding a rough value of g.

Ordinary spiral springs such as are used for weighing generally obey Hooke's Law, and may be used for this experiment.

96. Combination of two *S.H.* Motions.—Suppose a point describes a circle radius a with uniform angular velocity ω. Suppose A is the starting point, O the centre, P the position of the moving point after a time t, N the projection of P on OA.

Then $$\text{angle } AOP = \theta = \omega t$$
$$ON = x = a \cos AOP$$
i.e. $$x = a \cos \omega t \quad \ldots\ldots\ldots(1)$$

Let us suppose that the point N has in addition to this simple harmonic motion another motion, also simple harmonic, at right angles to OA. We should have such a motion if we imagined N to move as before along the paper to and fro along AA', while the paper itself moves backwards and forwards in a direction at right angles. Suppose the second motion is given by the relation

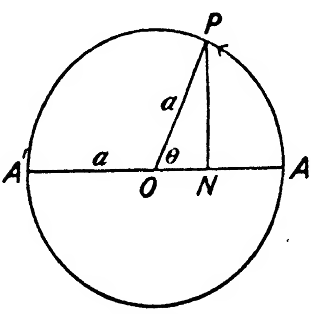


Fig. 47.

$$y = b \cos(\omega' t + \alpha) \quad \ldots.(2)$$

Here y is the distance of the moving point from the line AA', b is the amplitude of the second motion, and α the difference in phase between the two motions.

The relations (1), (2) suffice to define the positions of the moving points at any instant.

Let us take the particular case in which

$$\alpha = \frac{\pi}{2}, \ \omega = \omega'.$$

Then $$\frac{x^2}{a^2} = \cos^2 \omega t, \quad \frac{y^2}{b^2} = \sin^2 \omega t$$

so that $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This equation represents an ellipse.

If further $a = b$, the equation becomes $x^2 + y^2 = a^2$, and represents a circle.

Hence we may regard uniform circular motion as the combination of two similar simple harmonic motions at right angles to one another, which differ in phase by 90°. These motions are considered in the theory of circular polarisation of light.

97. Further Extension.—It can be shown that any periodic motion in a straight line may be regarded as the resultant of a number—finite or infinite—of simple harmonic motions; and, further, that any coplanar periodic motion whatsoever can be analysed into two sets of such motions directed along two lines at right angles.

Examples VII.

1. Show that there are four points collinear with the centre of gravity of a pendulum which are such that the times of oscillation about them are equal.

Suppose C is a point, distant x from the c.g. G of a pendulum, about which the time of oscillation is t. Then

$$t = 2\pi\sqrt{\frac{x^2+k^2}{xg}}$$

$$\therefore \quad x^2 - \frac{gt^2}{4\pi^2}x + k^2 = 0.$$

$$\therefore \quad x = \frac{gt^2}{8\pi^2} \pm \sqrt{\frac{g^2t^4}{64\pi^4} - k^2}$$

$$= a + b \text{ (say)} \quad \dots\dots\dots\dots(1)$$

$$\text{or} \quad a - b \quad \dots\dots\dots\dots(2)$$

If the pendulum be inverted we shall get these results repeated. Hence there are in all four points. Of these two (C_1, C_2, say) are on one side of G at distances $a + b$, $a - b$ from G, the other two (C_3, C_4) are on the other side at the same distances $a + b$, $a - b$.

C_1 and C_3 are therefore equidistant from the c.g. So also are C_2 and C_4. The distance between C_1 and C_4 or between C_2 and C_3 is

$$\overline{a+b} + \overline{a-b} = 2a = \frac{gt^2}{4\pi^2}.$$

This is the length of the simple equivalent pendulum.

2. A metre rod is hung up as a pendulum and allowed to swing. At what point must it be suspended to have the least possible period?

3. If the earth were a homogeneous sphere, and a straight frictionless tunnel could be constructed between any two places on the surface, prove that a train could run from one to another in a time independent of the distance apart of these places. Find this time.

4. A quantity of gas is enclosed in a cylinder in which works a smooth heavy piston. The axis of the cylinder is vertical. The piston is thrust down to compress the gas and then let go. Is the ensuing motion simple harmonic?

5. Borda's pendulum consisted of a spherical bob suspended by a fine wire. If the length of the wire from centre to point of suspension was 1 m. and the radius of the ball 2·5 cm., find the length of the simple equivalent pendulum.

6. Find how the energy of a simple pendulum depends on the amplitude.

7. Show how to find the resultant motion obtained by compounding together two equal uniform circular motions, of the same period, in the same plane:—(*a*) when the two motions are in the same sense; (*b*) when they are in opposite senses. What will be the resultant motion if the two circular motions are in opposite senses, and differ very slightly from one another in period?

8. Prove formula for the time of swing of a magnet in the earth's field, viz. $t = 2\pi\sqrt{\frac{K}{MH}}$.

9. What are the characteristics of the motion of a body which is performing isochronous vibrations?

10. What is a seconds pendulum? Why is it necessary that the arc of vibration of a pendulum should be small?

11. Calculate the moment of inertia of the balance wheel of a watch, assuming the wheel to consist of a brass rim 1 cm. in radius weighing 2 grammes, with weightless spokes. If the watch is accurate at 15° C., calculate approximately its gain or loss per day when kept at 0° C.; the coefficient of linear expansion of brass being 0·000019.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER VIII.

TIME.

98. The Apparent Motion of the Sun.—Every day the sun appears to rise, to travel round an arc in the sky, and to set again. At night certain fixed stars and constellations lie on or near the arc traced out by the sun in the day. These stars rise and set just as the sun does, tracing out the same arc. The sun, however, does not move at quite the same rate as the stars, and so appears to travel through a belt of stars, completing its full journey once a year. If we could see the stars in the daytime we should see the sun always in this belt. The constellations through which the belt passes comprise together the signs of the Zodiac. The path of the sun among the stars is the *ecliptic*. The moon, the planets and their moons, all lie very nearly in the same plane, the plane of the ecliptic.

99. The Celestial Sphere.—Imagine a sphere of great radius described with the same centre as the earth (Fig. 48). This is the celestial sphere. The lines joining up the heavenly bodies to the centre will cut this sphere in points. We may suppose that the bodies are actually set in the celestial sphere at these points of intersection. The plane of the equator (of the earth) cuts the celestial sphere in a circle also termed the equator. This circle is fixed, since the direction of the axis of the earth is fixed. The path of the sun on the celestial sphere is another circle, the plane of which is inclined at an angle of about $23\frac{1}{2}$ to the plane of the equator. The path is the ecliptic. The two circles cut at two points called the First Point of Aries ♈ and the

First Point of Libra ♎. The sun in its journey round the ecliptic is at these points on March 21, September 23 every year.

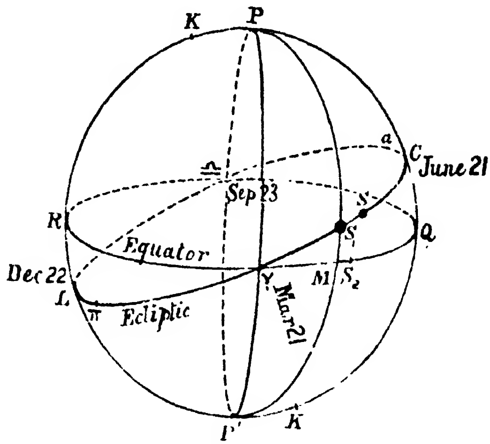


Fig. 48.

100. The Sidereal Day.—The earth turns round on its axis with a uniform angular velocity. Hence to an observer situated at the centre of the earth and turning with the earth, the celestial sphere would appear to rotate with uniform angular velocity. The period is called a sidereal day. Or we may define a sidereal day as the interval of time between the consecutive occasions on which any particular star is due North or due South of an observer on the surface of the earth, *i.e.* between consecutive southings of a star. The sidereal day is of constant length and so would serve as a unit of time; but it is not a convenient unit, as we obviously need a unit closely related to the day measured by the sun.

101. Perhaps the following explanation of the difference between solar and sidereal days may be serviceable.

Suppose (Fig. 49) that S represents a sun, P a planet moving round it in a circle. Let the planet rotate uniformly on its axis and its year contain exactly ten of its solar days. At any particular time let a point a be

directly opposite to the sun. Then after one solar day the same point will again be directly opposite, but the centre of the planet will have moved from o to o'. Let a' be the new position of a. The angle $o' S o = \frac{2\pi}{10}$. It is evident from the figure that the planet has really turned through an angle greater than four right angles by the angle marked θ; but

$$\theta = o' S o = \frac{2\pi}{10}.$$

Hence every solar day the planet turns through an angle $\quad 2\pi + \frac{2\pi}{10}$;

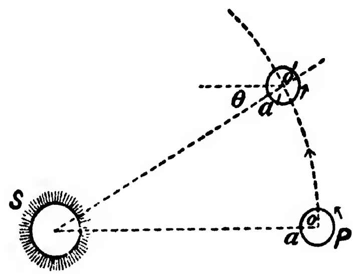


Fig. 49.

so that in the year of 10 solar days it turns through 10 times as much, *i.e.* it makes 11 complete turns, so that 11 sidereal days = 10 solar days.

102. Equation of Time.—The solar day is the period between two consecutive southings of the sun; if the sun moved with uniform velocity in the equator this period would be of constant length. The inclination of the ecliptic to the equator combined with the want of uniformity in the velocity of the sun causes the length of a

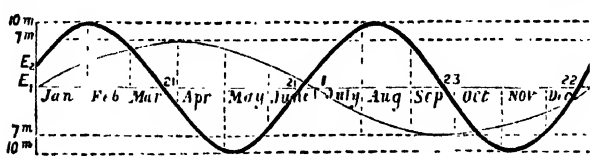


Fig. 50.

solar day to vary by a few seconds. It is due to this that the time registered by a sundial may differ from the Greenwich time. The difference is cumulative and mounts to about 16 minutes, and then diminishes again.

The reason that the sun does not travel with uniform velocity is that the path of the earth round the sun is not an exact circle, but an ellipse.

In Figure 50 the thin curve represents the portion of the equation of time due to unequal motion. The thick curve represents the portion due to obliquity.

In Figure 51 the combined effects of the two are shown.

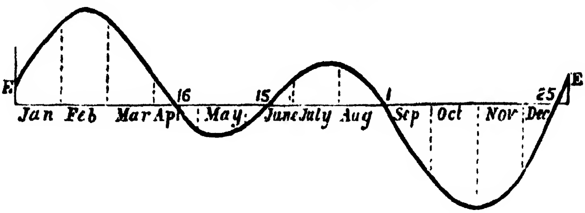


Fig. 51.

For times in the year in which the curve lies below the axis, the sundial is *before* the clock.

103. The Mean Sun.—To get a period of constant length governed by the motion of the sun, we imagine the motion of a point called the mean sun.

Imagine a point (S_1) (Fig. 52) to start from the first point of Aries (♈) with the sun, to travel round the ecliptic at a *uniform* rate and reach the first point of Libra ♎ with the sun. Such a point is called the dynamical mean

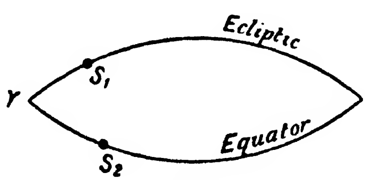


Fig. 52.

sun. Now imagine a second point S_2 to travel round the equator at such a rate that ♈S_2; measured along the ecliptic is always equal to ♈S_1 measured along the equator. This point is called the mean sun. The mean sun has uniform motion, and so serves to measure time.

104. The **Mean Solar Day** is defined as the interval of time between two consecutive southings of the mean sun: the interval between mean noon one day and mean noon the next. It is a period equal to the average length of a solar day.

The Second is the standard unit of time: it is equal to $\frac{1}{60} \times \frac{1}{60} \times \frac{1}{24}$ of a mean solar day.

$365\frac{1}{4}$ mean solar days $= 1$ year $= 366\frac{1}{4}$ sidereal days.

Examples VIII.

1. The ordinary year contains 365 days, the leap year 366 days. A leap year is one the number of which is divisible by 4, with the exception that years whose numbers end in 00 are only leap years if their numbers are divisible by 400.

Show that the average length of the year is 365·2425 days.

2. Find the ratio of the sidereal second to the solar second.

(For further examples on this chapter see Miscellaneous Examples, p. 258.)

CHAPTER IX.

SOLIDS.

105. Elasticity is a general name given to that property of a body in virtue of which it resists, and recovers from, change of shape or volume. All substances resist changes in volume, and so have what is termed bulk elasticity, but it is only solids that have elasticity of shape: no fluid—liquid or gas—can offer a permanent resistance to change of shape. On this property the definition of a fluid is based. The deformation, whether of shape or bulk, which a body experiences when forces are impressed upon it is termed *strain*. The equilibrating system of forces which produces a strain is termed a *stress*. The ratio of stress to strain is termed a *modulus* of elasticity. The subject of elasticity is one of great difficulty. We shall confine our work here to a few simple strains. For more advanced investigations Kelvin and Tait's *Natural Philosophy* or Love's *Elasticity* may be read.

106. Definitions.—A body is said to be *homogeneous* if two equal rectangular blocks cut from it, the edges of one parallel to the corresponding edges of the other, are exactly alike and indistinguishable from one another. Lead, wax, jelly, quartz, glass satisfy the conditions. A substance may be homogeneous to some tests and not to others. Thus well mixed mortar is homogeneous so far as a builder is concerned, but of very small portions taken from such a mixture one may be only sand, another water, and a third lime.

Now in some substances the properties of a brick-shaped portion depend on the direction in which it is cut. Wood is such a substance, for if two rods be cut one parallel to

the grain and the other across it, the one will be tough and strong, the other weak and brittle.

Quartz conducts heat better in one direction than in another. Hence bricks cut from the same crystal but differently orientated could be distinguished from one another. A sphere of quartz, if warmed, does not remain a sphere—it expands more in one direction than in another. Such a substance is termed *Aeolotropic*. A body which is such that two equal similar portions cut with any orientation are exactly alike and indistinguishable is termed *Isotropic*. Glass is such a substance. It is both homogeneous and isotropic. Quartz crystals are homogeneous, but aeolotropic.

107. Extension of a Rubber Cord.—Hang up a spiral spring or a piece of rubber beside a rule after the manner indicated in Fig. 53.

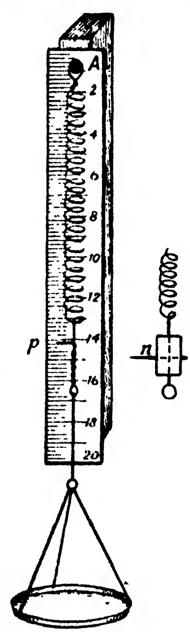


Fig. 53.

Measure the length of the rubber when not stretched—suppose this is l. Then put a small weight in the scale-pan and notice the extension produced in the rubber. Add another weight, and again note the extension. Always measure the extension by noting the position of the spike on the rule, subtracting the original reading. Proceed in this way, gradually increasing the load.

Compare each load with the total extension it produces. Plot the results on squared paper, the ordinate representing the load, the abscissa the pull. It will be found that at first, when the load is small, the rubber increases in length by the same amount for the same increase in the load. Thus if a 1 oz. weight produce an extension of 1 cm. an additional ounce will produce an additional stretch of 1 cm., so that for a 2 oz. weight the total extension will be 2 cm. In other words the total extension is proportional to the total

load. This result, however, is only true when the load is small.

108. Hooke's Law.—The relation connecting the extension with the load may be looked on as a particular example of Hooke's Law. This law—"*ut tensio, sic vis*"—states that stretching is proportional to the force producing it.

The law is true for many cases besides the stretching of elastic. Thus the bending of a rod, the deflection of a stretched fiddle-string, the compression of a gas are all proportional to the forces causing them provided that the strain produced is small.

109. Young's Modulus.—Suppose a wire of length L and cross-section a is stretched an amount l by a force F acting along its length. Hooke's Law states that $l \propto F$. Young modified Hooke's statement to—

Stress is proportional to strain,

and the ratio of longitudinal stress to longitudinal strain is called the *Young's Modulus* of the material of which the wire is composed. This modulus of elasticity is usually denoted by Y.

The longitudinal stress $= F/a$,
The longitudinal strain $= l/L$.

$$\therefore \; Y = \frac{F}{a} \Big/ \frac{l}{L} = \frac{F}{a}\frac{L}{l} \qquad \text{.......................(1)}$$

If we could imagine a wire of unit sectional area pulled out by a force to twice its natural length, then this force would measure *Young's Modulus* for the material. Of course in practice it is impossible to double the length of a wire without breaking it, and the law (1) ceases to hold even in the case of rubber long before the length of a cord could be doubled.

The dimensions of Y are

1 in Mass,
−1 in Length,
and −2 in Time.

110. Method of finding Young's Modulus.—The apparatus usually employed for finding *Young's Modulus* for a wire is shown in Fig. 54.

AB is the wire, fixed tightly at *A*, and carrying a scale-pan *P* which can be loaded as required.

Hanging close beside *AB* is a second wire *CD*. This is kept taut by a weight *W*. It carries a carefully graduated scale, *S*, which remains fixed during the experiment. Over this scale moves a vernier, *V*, which is carried by and fixed to the wire *AB*. As the load in *P* is increased, the length of the wire between *A* and *V* increases; the extension is measured by the position of the vernier on the scale.

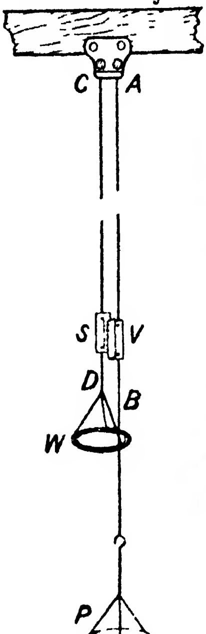


Fig. 54.

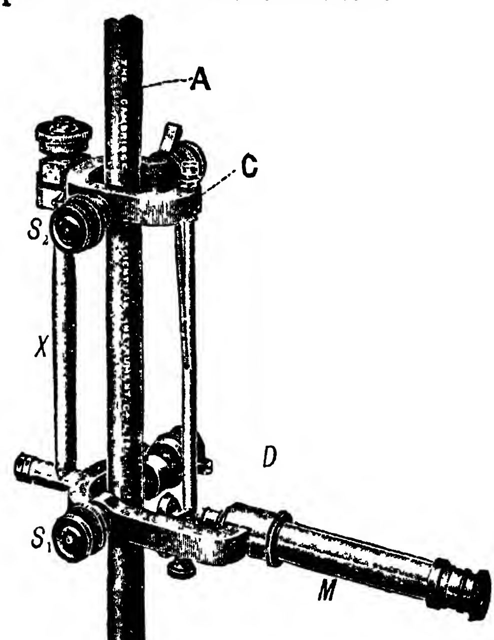


Fig. 55.

111. Ewing's Extensometer is a convenient apparatus for finding the modulus of specimens of metals given in the form of short rods or bars. In Fig. 55 the bar (*A*) to be tested is shown vertical. Two clamps are attached to it by the points of two set screws,

S_1 S_2. The lower clamp has a projection (X) which is parallel to the bar. The upper end of this is rounded and engages with a conical hole in the upper clamp. CD is a rod parallel to X. The axis of the test piece A is equidistant from the two rods X and CD. CD is free to slide in a guide on the lower clamp. A mark on it is viewed by the microscope M.

When A is stretched, this mark is seen to move in the field of view of the microscope. The distance is measured on a micrometer scale in the eye-piece. The extension of the bar A is half this distance.

112. Behaviour of a Strained Wire.—The relation between the extension of the portion $A\,V$ and the load in P may be plotted as in Fig. 56. At first the curve is straight, indicating that the tension is proportional to the

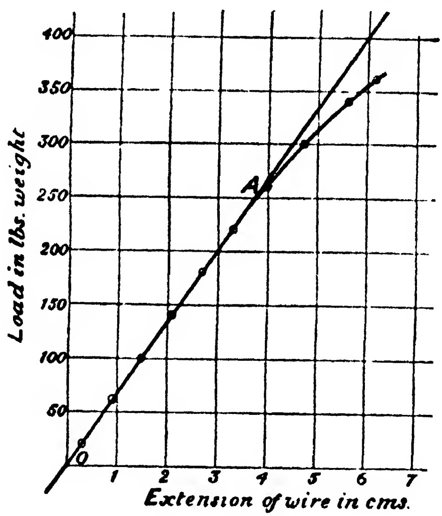


Fig. 56.

(The curve was obtained from a steel wire; diameter ·107 cm., length 6 cm.)

stretching, but in the neighbourhood of A the curve bends down: the wire has been stretched beyond its elastic limit. If the load had been removed before this point was reached, the wire would have recovered its natural length. After passing this point, however, the wire will not return entirely, but has acquired a permanent set.

When the load is still further increased the length begins to increase very fast for small increases in the load. The *Yield point* has been reached, the metal appears to *flow* under the forces imposed and the wire soon breaks.

During the first part of the extension represented by the straight portion OA of the curve the wire is said to have *perfect elasticity*. Up to this point it has the power of recovering its original length if unloaded. Some materials recover almost immediately; others take much longer. Steel appears to recover faster than brass. Glass is often very slow indeed and may take hours or days.

Another curious effect is exhibited by glass. If a fibre of glass be twisted first, say, to the right and kept there for some time, afterwards to the left for a short time, and then set free, it will twist back slowly to the right past the initial position and then slowly turn back again to the left, finally reaching the state in which it was before the experiment began.

113. Elastic Fatigue.—Another property exhibited by some solids is called elastic fatigue. Suppose two exactly similar torsion pendulums, A and B, are mounted side by side. Let one (A) remain at rest. Arrange for the other (B) to be kept twisting backwards and forwards for several days. Then start A and B both twisting through the same arc and notice the times they take to come to rest. It will be found that the time for B is less than for A. This form of experiment is due to Lord Kelvin. It shows plainly that the nature of the wire is dependent on its history. In this respect solids differ essentially from liquids and gases. A wire loaded gradually can sustain a greater weight than would be sufficient to break it if applied all at once.

It seems probable that bars of metal subjected to constant shaking and knocking become more brittle, and this has been given as a reason for the snapping of railway truck axles.

Crane chains are said to behave in a similar way. In engineering works it is usual to heat them up periodically in order that they may recover their normal strength.

114. Poisson's Ratio.—Suppose a rod, of unit cross-sectional area, is stretched by a force T and that the extension per unit length which it produces is aT. As the rod is pulled out its sectional area decreases by an amount proportional to T. Suppose the bar is square in section, and that the contraction of each side is βT. The ratio $\frac{\beta}{a}$ is often termed Poisson's ratio. The value of this varies

considerably, though Poisson was led to believe it was always $\frac{1}{4}$ in isotropic substances.

115. Deformation of a Cube.—Let $OXyZxYzo$ (Fig. 57) represent a cube. Let forces equal to T_z act on the faces $OXzY$, $oxZy$ perpendicular to them, producing an elongation $T_z\alpha$ in the edges parallel to OZ, and a contraction $T_z\beta$ in each of the edges OX, OY.

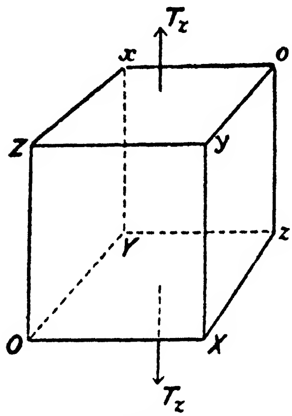


Fig. 57.

The lengths of the edges then become

$$OX = 1 - \beta T_z,$$
$$OY = 1 - \beta T_z,$$
$$OZ = 1 + \alpha T_z.$$

Let similar force T_x parallel to OX be now applied to the faces $OZxY$, $ozXy$. These will produce extensions parallel to OX, and contractions parallel to OY, OZ, and the lengths of OX, OY, OZ will become

$$OX = 1 - \beta T_z + \alpha T_x,$$
$$OY = 1 - \beta T_z - \beta T_x,$$
$$OZ = 1 + \alpha T_z - \beta T_x.$$

If now a third pair of forces T_y act on the faces $OXyZ$, $oxYz$, the lengths will become

$$OX = 1 - \beta T_z + \alpha T_x - \beta T_y,$$
$$OY = 1 - \beta T_z - \beta T_x + \alpha T_y,$$
$$OZ = 1 + \alpha T_z - \beta T_x - \beta T_y.$$

Hence the volume of the cube becomes equal to the product of these three quantities, *i.e.* to

$$1 + (\alpha - 2\beta)(T_x + T_y + T_z)$$

if we neglect squares and products of the small quantities α, β.

116. The Bulk Modulus, k.—In the particular case in which $T_x = T_y = T_z = T$ this result shows that the increase in volume is $3T(\alpha - 2\beta)$.

If the forces applied, instead of being tensions pulling out the cube, are thrusts tending to compress the cube, the increase of volume may be found by reversing the sign of T. Hence a cube of unit volume would under a hydrostatic pressure P contract by an amount $3P\ (\alpha - 2\beta)$. Now the contraction per unit volume per unit increase of pressure is termed compressibility. Hence the compressibility of a substance is measured by the expression $3(\alpha - 2\beta)$. Its reciprocal is termed the *bulk modulus of elasticity*, and is denoted by k: hence $\frac{1}{k} = 3(\alpha - 2\beta)$.

117. Modulus of Rigidity.—Let $ABCD$ (Fig. 58) represent an end view of a cubical block, the edge of which is of unit length. Call the face parallel to this $A'B'C'D'$. Suppose a couple T, T act on the faces $ABB'A'$, $DCC'D'$, and that the forces of the couple are in the directions shown.

On the faces $ADD'A'$, $BCC'B'$ let the forces of another equal couple act. These couples will be in equilibrium with one another, but will cause the cube to be deformed so that the faces $ABCD$, $A'B'C'D'$ become rhombuses.

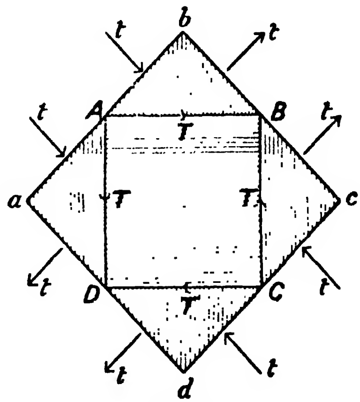


Fig. 58.

Suppose this causes the angle at D to decrease by an amount θ. Then the ratio $T \div \theta$ is termed the *modulus of the rigidity* of the substance. In other words the modulus of rigidity is defined as the ratio of the tangential force per unit area to the angular deformation produced. The rigidity is generally denoted by n.

$$n = T \div \theta.$$

118. The Value of n in Terms of α, β.—The means by which the tangential forces T have been applied have not been specified, and are immaterial so far as the deformation of the block is concerned. We may imagine then that these forces are applied by means of four blocks glued on to the four faces $ABB'A'$, $BCC'B'$, $CDD'C'$, $DAA'D'$ of the cube. An end view of these blocks is shown vertically shaded in Fig. 58. The whole figure then makes up a square, $abcd$, the sides of which are parallel to the diagonals of the square $ABCD$.

Call the other corners of the complete block $a'b'c'd'$. Consider the prism-shaped block $ABbb'B'A'$. Suppose forces t, t act on the faces $Abb'A'$, $bBB'b'$. These are indicated in the figure. The resultant of these two forces is $2t \cos 45°$, *i.e.* $t\sqrt{2}$, acting parallel to AB. Hence if $T = t\sqrt{2}$ we may regard the tangential force T on the face $ABB'A'$ as being produced by the forces t, t on the faces of the prism. Treating the other portions in a similar manner, we see that the four forces T may be supposed due to the eight forces t, acting as shown in the figure.

We shall now calculate the deformation produced by the eight forces. The length of AB is unity: therefore the length Ab is $\dfrac{1}{\sqrt{2}}$, and the area of the face $Abb'A'$ is $1 \times \dfrac{1}{\sqrt{2}}$. The force acting on this is t or $\dfrac{T}{\sqrt{2}}$; hence the force per unit area acting on this face is $\dfrac{T}{\sqrt{2}} \div \dfrac{1}{\sqrt{2}}$, *i.e.* is T. The thrust per unit area on the whole face $abb'a'$ is therefore T.

The deformation of the cube $ABCD$ is therefore that which would be produced were it the centre portion of a block on the faces of which thrusts or tensions of T per unit area are acting.

Now by Art. 115 these forces would produce extensions of $T(\alpha + \beta)$ per unit length in ab and cd, and contractions of $T(\alpha + \beta)$ per unit length in bc, da. These results are obtained by putting $T_x = T = - T_y$, and $T_z = 0$ in the equations. Now the length of side of $abcd$, *i.e.* the length of the diagonals AC, BD of the faces of the cube, is $\sqrt{2}$.

Hence under the action of the forces considered, the diagonal AC will shorten by an amount $T(\alpha+\beta)\sqrt{2}$, while BD will be pulled out by an equal amount.

The ratio of the diagonals

$$= \frac{\{1 - T(\alpha+\beta)\}\sqrt{2}}{\{1 + T(\alpha+\beta)\}\sqrt{2}}$$

$$= \frac{1 - T(\alpha+\beta)}{1 + T(\alpha+\beta)}.$$

Suppose Figure 59 represents a section of the deformed cube. With the notation of Art. 117 the angle ADC is less than a right angle by θ.

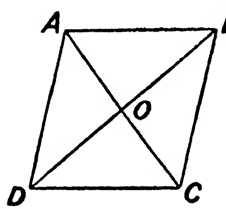


Fig. 59.

Hence $$ODC = \frac{\pi}{4} - \frac{\theta}{2},$$

$$\therefore \frac{1 - T(\alpha+\beta)}{1 + T(\alpha+\beta)} = \frac{AC}{BD} = \frac{OC}{OD} = \tan ODC$$

$$= \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}$$

$$\therefore T(\alpha+\beta) = \tan\frac{\theta}{2} = \frac{\theta}{2} \text{ approximately};$$

but $$n = \frac{T}{\theta},$$

$$\therefore n = \frac{1}{2(\alpha+\beta)}.$$

119. Young's Modulus in Terms of k, n.—If a cube of unit edge be acted on by a unit tension parallel to one of the edges the extension produced is α, and we have (Art. 109) $1 = Y\alpha$, where Y is Young's modulus. We have then the three relations

$$(1) \qquad \alpha = \frac{1}{Y}$$

$$(2) \qquad \alpha + \beta = \frac{1}{2n} \qquad \text{(Art. 118.)}$$

$$(3) \qquad \alpha - 2\beta = \frac{1}{3k} \qquad \text{(Art. 116.)}$$

Solving (2) and (3) for α and β we get

$$3\beta = \frac{1}{2n} - \frac{1}{3k}, \quad i.e. \quad \beta = \frac{3k - 2n}{18nk},$$

$$3\alpha = \frac{1}{n} + \frac{1}{3k}, \quad i.e. \quad \alpha = \frac{n + 3k}{9nk}.$$

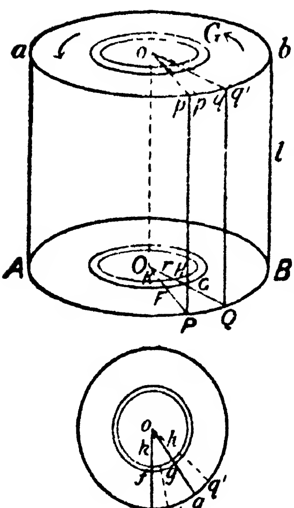


Fig. 60.

Hence from (1) we get

$$Y = \frac{9nk}{n + 3k},$$

an expression for Young's modulus in terms of the moduli of rigidity and bulk.

120. Twisting of a Cylinder.—Let $ABba$ (Fig. 60) represent a short cylinder of metal. Call the length of the cylinder l. Let P, Q be points close together on the rim AB, and p, q the corresponding points on the rim ab.

We shall suppose that the base AB of the cylinder remains fixed, while the top ab is twisted round the axis Oo through a small angle θ, by a couple G. Let p', q' be the new positions into which p, q are twisted. Then the angle $pop' = \theta = qoq'$.

With centre O let two circles, radii r, $r+dr$, be described on the base AB, cutting OP, OQ in $FGHK$. Let the parallel circles described on the top cut op, oq in $f\ g\ h\ k$ and op', oq' in $f'\ g'\ h'\ k'$. Then the small rod $FGHKfghk$ (Fig. 61) is twisted into the shape $FGHKf'g'h'k'$. The angle

$$fFf' = \frac{ff'}{fF} = \frac{r\theta}{l}.$$

Suppose the twisting force applied to the top is S per unit area. Then if n is the rigidity, we have (Art. 117)

$$S = n \, . \, \frac{r\theta}{l}.$$

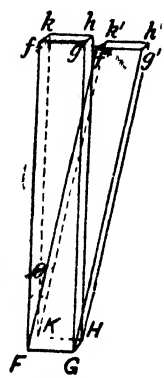


Fig. 61.

Let the area of the face $fghk$ be denoted by a. Then the tangential pull on this area is

$$Sa = n\frac{r\theta}{l}a,$$

and the moment of this about the axis is

$$Sar = \frac{n\theta}{l} \, . \, r^2a.$$

Now we may regard the whole cylinder as divided into little rectangular rods like $FGHKfghk$, and the top as divided up into little blocks like $fghk$.

The sum of the moments round the centre of the forces acting on the top will be

$$\Sigma Sar = \Sigma\frac{n\theta}{l}r^2a$$

$$= \frac{n\theta}{l}\Sigma r^2a.$$

But Σr^2a is the moment of inertia of a disc of unit density about its centre; hence $\Sigma r^2 a = \pi a^2 \, . \, \frac{a^2}{2}$, where a is the radius of the top.

Hence the sum of the moments $= \frac{n\theta}{l} \, . \, \frac{\pi a^4}{2}.$

Now the sum of the moments of these forces is the moment of the twisting couple, G. Hence

$$G = \frac{n\theta}{l} \cdot \frac{\pi a^4}{2}.$$

121. Torsion of a Wire.—In the preceding article we regard the cylinder as being short. We do this in order that the angle which a generating line when deflected (pP, Fig. 60) makes with the axis may be regarded as small. We may regard a long wire as built up of a large number of short cylinders, and the result

$$G = \frac{n\theta}{l} \cdot \frac{\pi a^4}{2}$$

will still hold; moreover in this case the angle θ through which one end is twisted relatively to the other need not be small: it is only necessary that the angle corresponding to pPp' be small.

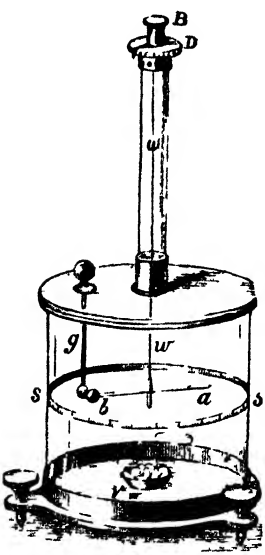


Fig. 62.

Hence if a wire of radius a be fixed at one end and acted on by a couple G at the other, causing the generating lines to be twisted into spirals tangents to which are inclined at small angles to the axis, then

$$G = \frac{n\theta}{l} \cdot \frac{\pi a^4}{2},$$

when l is the total length of the wire and θ the angle through which one end is twisted.

122. Torsion Balance.—From this relation we see that $G \propto \theta$. This means that if a wire be twisted by a couple the torque or twist produced is proportional to the moment of the couple. The result was used by Coulomb in his torsion balance for measuring the repulsion between two electric charges. The balance consists of a long wire, w

(Fig. 62), suspended from a brass head, B, carrying at the lower end a rod ab. B can be twisted round in its support D. On D is a scale which shows the angle through which the upper end of the wire is twisted. By means of B the two balls at b may be brought close together or separated. The twist on the wire can be measured by the scales D and SS.

The lower scale, SS, gives the angle through which the bottom of the wire has turned, the upper, D, that through which the top has turned. The sum or difference of these two gives the total twist on the wire. The force between the two balls at b when separated by different distances can thus be compared.

We shall have occasion to refer to the torsion balance subsequently (Art. 164, 166).

123. Torsion Pendulum.—The fact that the couple exerted by a twisted wire is proportional to the angle of twist enables us to get simple harmonic vibrations of which the amplitude may be considerable. In Fig. 63 AB is a wire, DEF a disc of metal, the centre of which is at B. A is fixed. The end B twists with the disc.

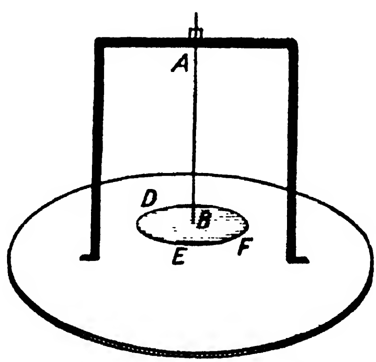


Fig. 63.

Suppose that originally the disc was displaced from its equilibrium position by twisting it through an angle a, and that the angle of twist at any subsequent time, t, is θ.

The moment of the torsion couple

$$= G = \frac{n\theta}{l} \cdot \frac{\pi a^4}{2}$$

$$= c\theta \text{ say}$$

$$\text{if} \quad c = \frac{n\pi a^4}{2l}.$$

Suppose the disc is made up of little masses $m_1, m_2, m_3, \ldots$ at distances $r_1, r_2, r_3, \ldots$ from the centre B. These have all the same *angular* acceleration ϕ (say). Hence their acceleration along the tangents to their path $= r_1\phi, r_2\phi, r_3\phi, \ldots$ the forces producing these accelerations are $m_1r_1\phi$, $m_2r_2\phi, \ldots$ and their moments round the centre $m_1r_1^2\phi$, $m_2r_2^2\phi, \ldots$ and hence the sum of their moments round the centre

$$\begin{aligned}&= m_1r_1^2\phi + m_2r_2^2\phi + \ldots\\ &= (m_1r_1^2 + m_2r_2^2 + \ldots)\phi,\\ &= \phi\Sigma mr^2,\\ &= \phi Mk^2,\end{aligned}$$

where Mk^2 is the moment of inertia of the disc about its centre. The sum of these moments must be equal to the moment of the couple G.

Hence $$c\theta = Mk^2\phi,$$

or $$\phi = \frac{c}{Mk^2}\theta.$$

Hence the angular acceleration is proportional to the angle of twist, *i.e.* angular acceleration $= \frac{c}{Mk^2} \times$ angular displacement. Moreover the angular acceleration is always in the direction tending to restore the disc to the equilibrium position.

The motion is therefore simple harmonic and of fixed period, T, where

$$T = 2\pi\sqrt{\frac{Mk^2}{c}},$$

i.e. $$T = 2\pi\sqrt{\frac{Mk^2 \cdot 2l}{n\pi a^4}},$$

or $$n = \frac{8\pi l \cdot Mk^2}{T^2 \cdot a^4}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $l =$ length, $a =$ radius, $n =$ the rigidity of the wire, $T =$ the period, and Mk^2 the moment of inertia of the disc.

124. Experimental Method of Finding Rigidity.—A convenient apparatus for finding the rigidity of the material of a wire is illustrated in Fig. 64.

The wire experimented on passes down the vertical tube and is fixed at the top end. The lower end is firmly attached to the cylinder. Round this cylinder pass two strings carrying scale-pans. As loads are placed in the pans the cylinder twists the wire. The twist on the wire is shown by the pointer which passes over the graduated circle.

Let us denote the radius of the cylinder by r, the load (including the scale-pan) fastened to each string by m. Let this load twist the cylinder through an angle θ. Then the moment of the couple exerted by the strings on the cylinder is $2\,mgr$.

This couple is balanced by the couple exerted by the wire,

$$\therefore\ 2\,mgr = G = \frac{n\theta}{l} \cdot \frac{\pi}{2} a^4,$$

if we employ the notation of Art. 121; *i.e.*

$$n = \frac{4\,mgrl}{\pi \theta a^4} \quad \ldots (2).$$

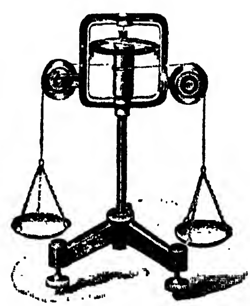


Fig. 64.

Equation (2) shows that for the same wire $\theta \propto m$. Hence if the relation connecting θ and m be plotted on squared paper, the result should be a straight line. The actual result will show to what extent the wire obeys the law "stress varies as strain," and how far the wire may be considered perfectly elastic.

In the result (2) θ is the circular measure of the angle of twist. To convert degrees to circular measure, multiply the number of degrees by $\frac{\pi}{180}$.

In using the result $n = 4\frac{m}{\theta} \cdot \frac{grl}{\pi a^4}$ to calculate n, the value of $\frac{m}{\theta}$ should be taken from the plotted curve, not from any particular values found.

The same apparatus may be used to show the isochronism of the torsion pendulum. To do this remove the strings, twist the cylinder from its equilibrium position through a small angle and set free. Find the time of swing. Repeat the operation with different initial twists. The period will be found invariable.

When the period has been found use equation (1) of Art. 123 to find the rigidity of the material, and so verify the result already obtained. The moment of inertia of the cylinder must be found by measuring the diameter, weighing, and applying the results given in Art. 76.

125. Spiral Spring.—The effect of pulling out a helical wire spring is practically to twist the wire round its own axis. This may readily be seen by examining the hair spring of a watch. If the spring rests in its usual position as a flat spiral, and one end be then lifted up while the other remains fixed, the spring becomes twisted. The stretched wire in a helical (or spiral) spring is in a state intermediate between that of a watch spring, which is bent but not twisted, and that of a torsion wire which is twisted but not bent. The stretching of an ordinary helical spring therefore brings into play forces of torsion. The amount of the torsion is proportional to the stretching, and also to the forces called into play; hence the extension of a helical string is proportional to its tension. The result, of course, only holds for small deformations. This is the principle on which ordinary spring balances are built.

126. Jolly's Balance consists of a long spiral spring (S) made in fine wire. This is held vertical, and behind it is a stand with a mirror scale, *MM*. From the bottom of the spring hangs a light scale-pan, Q. Readings of the extension of the spring are taken by holding the eye on the same level as some definite mark (P) at the bottom of the spring. When the eye is in this position P just hides

its own image in the mirror. The graduation on the mirror scale which is on the same level as P can then be read off.

Suppose the balance is required to weigh a mass (m) which lies between 5 and 6 grammes. Put a 5 gramme weight in the pan, and note the reading of P. Suppose it is a cm. Add an extra gramme and let the reading become b cm. Now replace the weights by the mass m and let the reading become c cm. Now a one gramme weight produced an extra extension of $(b-a)$ cm., therefore the weight which produced an extension of $(c-a)$ cm. was $\frac{c-a}{b-a}$ grms.

Hence the weight of m is

$$\left(5+\frac{c-a}{b-a}\right) \text{ grms.}$$

The balance is perhaps not of great practical use, but it at any rate dispenses with the use of small weights.

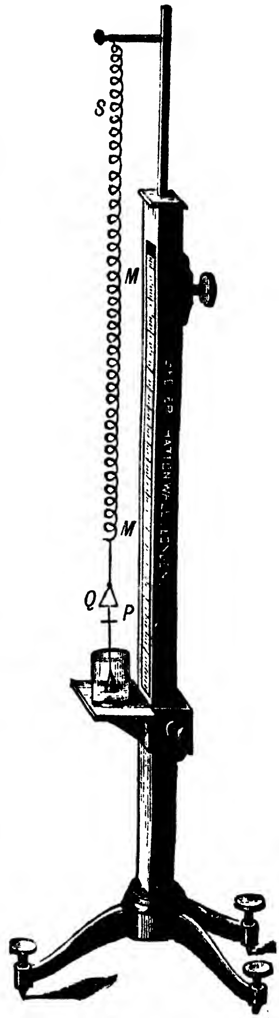


Fig. 65.

127. Impact.—If a round ball of clay or putty falls on the floor, it flattens itself out and does not rebound. A tennis ball also flattens itself out on the floor, but will rebound. The difference between the cases is due to the fact that the tennis ball not only resists change of shape, but has the power of quickly recovering its shape. The period during which the ball is in contact with the floor may be divided into two parts, one in which the ball is being more and more deformed, the other in which it is recovering its former shape. The forces exerted in the recovery are always less than those called into play during the earlier portion;

in consequence of this the velocity of rebound is less than the velocity with which the ball strikes the floor.

The laws of impact were investigated by Newton, who used apparatus similar to that shown in Fig. 66. The two balls A, B hang side by side, just touching each other when the strings are vertical. One of them, B say, is drawn back to the position B' and is then let go. At the bottom it strikes A. If B is large and A is small, then B will continue to move forward knocking A away in front of it. But if B is small and A large, then B will rebound off A, and A will move slowly forward.

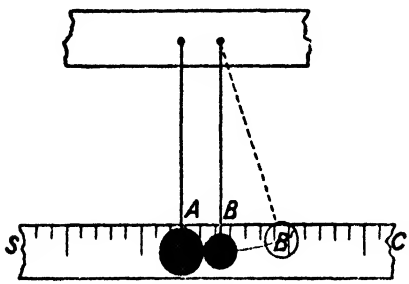


Fig. 66.

The velocity with which A starts off after the impact is proportional to the square root of the height to which it rises, and can therefore be calculated ($v^2 = 2gh$). The velocity of B just before impact and the velocity just after can be found in the same way. To find these velocities it is usual to measure the horizontal distances through which the balls travel. These distances are measured by placing behind A, B a mirror scale. From these distances the vertical fall can be found (see Example 1, p. 124).

The laws obeyed are—

(1) There is no change of momentum due to the impact —the momentum lost by B is equal to that gained by A.

(2) The relative velocity of the balls after impact bears to the relative velocity before impact a ratio which is independent of the initial velocities.

128. Conservation of Momentum.—The first law is general, and holds, whatever be the action between two bodies, it is deducible from the laws of motion. The action of B on A is always equal and opposite to the action of A on B. (Law III.)

The change of momentum produced by B on A is therefore exactly equal and opposite to the change in momentum produced by A on B, for change of momentum is proportional to the force (or action) producing it (Law II.). The total change of momentum is therefore zero. Newton's laws therefore lead up to the general principle known as the Conservation of Momentum, which may be stated thus: If bodies in any isolated system act on one another in any way, the total momentum of the system is unaltered.

129. Coefficient of Restitution.—The ratio of the relative velocity after impact to the relative velocity before impact is termed the coefficient of restitution, or resilience, or elasticity. We shall use the first term only. This coefficient is usually denoted by e. Newton's experiments showed that the value of e was, for fairly wide ranges, independent of the velocities. It differs considerably for different substances, being nearly 1 in the case of glass balls, nearly 0 for balls of clay. It is, however, always a proper fraction.

130. The Equations of Motion.—Suppose two balls of masses m, m' (Fig. 67) moving with velocities u, u' collide directly with one another, and that after impact their velocities are v, v'. Then if e is the coefficient of restitution, the laws of Article 127 give us

$$mu + m'u' = mv + m'v' \quad \text{.................} (1)$$

$$-e(u - u') = v - v' \quad \text{.....................} (2)$$

From these equations we can find v, v' in terms of m, m', u, u', and e. It should be noted that in these equations all velocities are measured in the same direction. This is indicated in the figure.

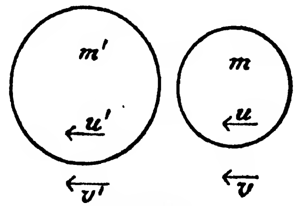


Fig. 67.

Should the numerical values of u, u' be given as, say, 5, 7 feet per second and the balls be travelling in opposite directions, then we choose one direction as positive, say that of the first, and write -7 for u in equations (1), (2).

131. Loss of Energy.—The kinetic energy of the balls before impact is $\frac{1}{2}mu^2 + \frac{1}{2}m'u'^2 = E$ say,
and the energy after impact is

$$\frac{1}{2}mv^2 + \frac{1}{2}mv'^2 = E' \text{ say.}$$

Combining these equations with those of Art. 130, we can deduce the result

$$E - E' = \frac{1}{2}(1 - e^2)\frac{mm'}{m + m'}(u - u')^2.$$

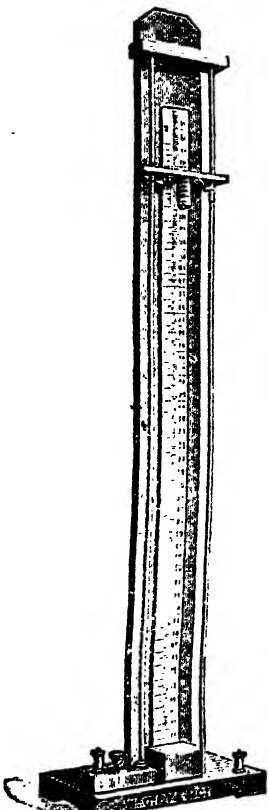


Fig. 68.

This expression is always positive because e is always less than unity. Hence kinetic energy is always lost by impact. It appears again as heat energy, and illustrates the principle laid down in Art. 63 that whatever action takes place available energy is always dissipated and degraded to heat.

132. Experimental Methods of Finding e.—A convenient way of finding e is to drop a small ball from a known height on a large horizontal block and note the height of rebound. Suppose the two heights are h, h', then the velocity just before impact was $\sqrt{2gh}$, the velocity after impact $\sqrt{2gh'}$.

Hence $e = \sqrt{\frac{2gh'}{2gh}} = \sqrt{\frac{h'}{h}}$.

A convenient apparatus is shown in Figure 68.

133. Temperature Effects.—The elasticity of a substance is dependent on its temperature. This is easily shown in the case of india-rubber. If a weight is suspended from a

rubber cord, the cord contracts if the temperature rises and lengthens if the temperature falls. This experiment has been said to show that rubber contracts when heated, but the real explanation is that rubber is less extensible when hot than it is when cold.

A necessary consequence of this property is that rubber if suddenly stretched rises in temperature; when allowed to contract its temperature falls. We shall not give a formal proof of the connexion of these two properties.

Let us assume, however, the principle that "an unconstrained system can never yield less to impressed forces than a constrained one." If the rubber conducts heat perfectly it always remains at the temperature of its surroundings; if it does not conduct heat at all it is to that extent constrained. It follows then from the principle enunciated that if temperature produces any effect at all, a rubber cord must yield less to a pull when stretched adiabatically than when stretched isothermally. Hence any temperature change caused by a pull increases the elasticity.

134. Bending of Rods.—If a straight rod is bent, the fibres on one side are lengthened and put in tension, while those on the other side are shortened and compressed. Somewhere in the rod is a surface such that the particles in it are at their normal distance apart. This surface is called the *neutral surface*; in the unstrained position of the rod it becomes a plane surface. In a symmetrical rod the neutral surface is perpendicular to the plane of bending, and it will be shown later that the longitudinal axis of this surface, called the **neutral axis**, passes through the centre of surface * of each cross-section of the rod.

It is evident that in every rod the amount of flexure produced by a system of applied forces must be dependent on the elasticity of the material of the rod. A rigid treatment of the subject is beyond the scope of this book, but the problem may be solved approximately by easy methods.

* Some writers call this point the centre of gravity of the cross-section; but, since its existence and position are independent of gravity, it is better to call it the "centre of surface"; just as in most cases it would be better to call the "centre of gravity" of a body its "centre of mass."

135. Forces in a Rod supporting a Mass at one End.—

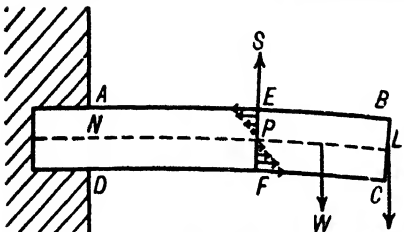


Fig. 68*a*.

Let $ABCD$ (Fig. 68*a*) represent a medial section of a rod fixed well into the wall at AB and supporting at the free end a mass M. Take a cross-section at EF. Let us now find the forces which keep the part $EBCF$ in equilibrium. A force Mg acts downwards from the end; therefore, acting upwards at the section EF, there must be a shearing force S equal to the sum of Mg and the weight of the portion of rod $EBCF$. These forces form a couple, and must be balanced by another couple. The only other forces which act on $EBCF$ are the forces due to the stretching and compression of the filaments of the rod which pass across EF. The filaments in the upper portion of the rod are in tension; the portion of a filament to the left of EF therefore exerts a pulling force on the portion to the right of EF. Similarly, a portion of a filament in the lower half of the rod, and to the left of EF, exerts a pushing force on the portion to the right of EF. These forces are distributed as shown in the figure, and their resultant is a couple which balances the couple formed by $Mg + W$ and S. This couple is called the **Bending Moment** at the section EF.

To find the point P, which is neither in tension nor compression—*i.e.* to find where the neutral surface cuts the section EF—take a little area a at a distance y above P. It is easily proved that the strain at a is proportional to y. Thus the tensile stress on a is proportional to y, and the force on a may be expressed by kay, where k is a constant. The sum of the forces acting to the left is equal to the sum of the forces acting to the right; therefore, counting tensile forces as positive and compressive forces as negative, we get $\Sigma kay = 0$, from which it follows that $\Sigma ay = 0$, *i.e.* P is at the centre of surface of the area EF. The neutral surface therefore passes through the centre of surface of all

cross-sections of the rod, and the line in the neutral surface joining all these centres of surface is the neutral axis. (See Art. 134.)

Let F denote the force on an area a situated at a distance y above or below the neutral surface. Then, taking moments around a line drawn through P perpendicular to the plane of bending, and denoting the bending moment at EF by B, we get

$$B = \Sigma Fy = \Sigma kay^2 = kI$$

where I is the moment of inertia of the section EF about the line drawn through P perpendicular to the plane of bending.

135a. Relation between the Bending Moment at a point and the Curvature of the Neutral Axis.—Let $ABCD$ (Fig. 68*b*) be a medial longitudinal section of a bent rod, and EF, GH the traces of two cross-sections near together and perpendicular to the neutral axis. Since the rod is bent, EF and GH will meet at O, which is therefore the centre of curvature of the portion PQ. Through Q draw KQM parallel to EF.

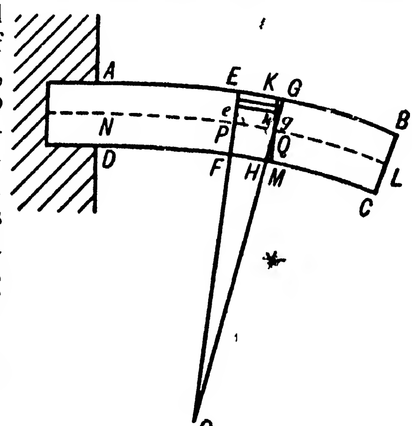


Fig. 68*b*.

Consider a filament of the rod parallel to and at a distance y from the neutral axis, and of original length ek and cross-section a. Its length now $= eg$, the elongation is therefore kg. Let $Y =$ the *Young's Modulus* of the material of the rod, then

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{\text{Force stretching the filament }/a}{kg/ek}$$

$$= \frac{\text{Force}/a}{kg/PQ} = \frac{\text{Force}/a}{y/OP} = \frac{\text{Force}\,.\,R}{a\,.\,y}$$

$\therefore$ Force stretching the filament $ek = \frac{Yay}{R}$.

$\therefore$ Moment of this force about $P = \frac{Yay^2}{R}$.

$\therefore$ Moment of all the horizontal tensile and compressive forces acting perpendicular to section EF, about a line through P perpendicular to the plane $ABCD$

$$= \Sigma \frac{Yay^2}{R} = \frac{Y}{R} \Sigma ay^2 = \frac{YI}{R}$$

where I is the moment of inertia of the cross section about a line through P perpendicular to the plane $ABCD$.

Denote the bending moment by B, then

$$B = \frac{YI}{R}.$$

If we neglect the weight of the part of the rod beyond the section in comparison with the load at the end, we can express B as $Mg(l - x)$ where M is the mass supported at the end, l is the length of the rod and x is the distance of the section from the support at the fixed end. We thus get

$$Mg\,(l-x) = \frac{YI}{R}.$$

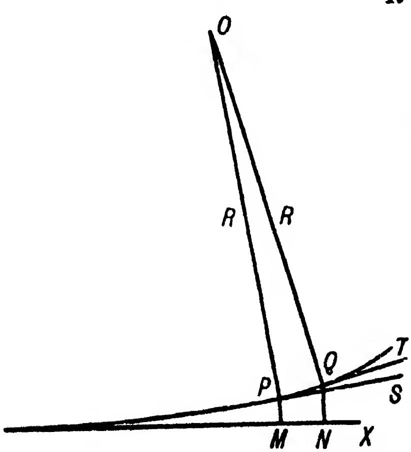


Fig. 68c.

135b. Expression for the Radius of Curvature of a Flat Curve in terms of the slope of the curve.—Let APQ (Fig. 68c) be a flat curve, P, Q two points on it near together, and AX, PS, QT tangents at A, P, Q. Let O be the centre and R the radius of curvature of the portion PQ of the curve.

Then, since the angle POQ is small, we have $R . \angle POQ = PQ$.

$R \times$ (difference in slope of tangents PS, QT) $= PQ$.

$$\therefore\ R\left(\frac{dy}{dx}\text{ at }Q-\frac{dy}{dx}\text{ at }P\right)=PQ=MN\quad\text{very nearly}$$

where MN is the projection of PQ on AX.

Denote MN by δx. Then when P and Q are very near together we have

$$\frac{1}{R}=\frac{\frac{dy}{dx}\text{ at }Q-\frac{dy}{dx}\text{ at }P}{\delta x}=\text{the value of }\frac{d^2y}{dx^2}\text{ at }P,$$

$$\text{since }\frac{dy}{dx}\text{ is small.}$$

135c. Values of the Deflections obtained in special cases of Bended Rods.—The equation obtained in the last article applies to the neutral axis in the case of bent rods in all cases where the curve of the axis is very slight.

The equation of Art. 135a now reads:

$$Mg(l-x)=YI\,.\frac{d^2y}{dx^2},$$

the axis of x being taken horizontal and the axis of y vertically downwards.

Case I. *Rod fixed horizontally at one end.*—Let length of rod exposed from support $=l$, and load supported at the end $=M$. Take a section at a distance x from the support at the fixed end. Then we have (by Arts. 135a and 135b)—

$$Mg(l-x)=YI\frac{d^2y}{dx^2};$$

$$\text{or, }\frac{YI}{Mg}\frac{d^2y}{dx^2}=l-x.$$

Integrating this we get

$$\frac{YI}{Mg}\frac{dy}{dx}=\int(l-x)dx=lx-\frac{x^2}{2}+\text{a constant.}$$

To find the constant we note that at A (Fig. 68*a* or *b*) $\frac{dy}{dx}=0$. The constant is therefore zero.

$$\therefore\frac{YI}{Mg}\frac{dy}{dx}=lx-\frac{x^2}{2}.$$

Integrating again we get

$$\frac{YI}{Mg}y = \int\left(lx - \frac{x^2}{2}\right)dx$$

$$= \frac{lx^2}{2} - \frac{x^3}{6} + \text{a constant.}$$

To find the constant we note that $y = 0$ when $x = 0$, ∴ the constant vanishes.

$$\therefore \frac{YI}{Mg}y = \frac{lx^2}{2} - \frac{x^3}{6}.$$

To find the depression of the end of the rod, put $x = l$, then denoting the depression of the end by δ we get

$$\delta = \frac{Mg}{YI}\left\{\frac{l^3}{2} - \frac{l^3}{6}\right\} = \frac{Mgl^3}{3YI}.$$

In the case of a rectangular rod of breadth b and depth d the area of cross-section $= bd$ and its moment of inertia about a line through the neutral axis perpendicular to the plane of bending $= bd \times \frac{d^2}{12} = \frac{bd^3}{12}$,

$$\therefore \delta = \frac{Mgl^3}{3Y.bd^3/12} = \frac{4Mgl^3}{Y.bd^3}.$$

Case II. *Rod supported at its ends on knife-edges.*—Let l = length of rod between supports and M = the load supported. In this case (Fig. 68*d*), if we regard, say, the left half of the rod as embedded in an immovable support, the right half behaves exactly as the rod in Case I., the force at the end being now upward and equal to $Mg/2$. The depression of the middle is therefore given by

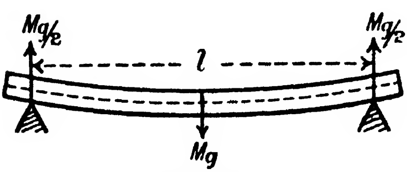


Fig. 68*d*.

$$\delta = \frac{\left(\frac{Mg}{2}\right)\left(\frac{l}{2}\right)^3}{3YI} = \frac{1}{48}\frac{Mgl^3}{YI}.$$

If the rod is rectangular in section, with breadth b and depth d

$$\delta = \frac{1}{48} \cdot \frac{Mgl^3}{Ybd^3/12} = \frac{Mgl^3}{4Ybd^3}.$$

135d. Determination of Young's Modulus from the Bending of a Rod.—A rod, AB (Fig. 68e), is mounted on two strong knife-edges, K K, placed a suitable distance apart. Standing on the mid-point of the rod is a small mm. or half-mm. scale, which is kept in position by a suitable balancing attachment, as shown in the figure. Weights are placed in a scale-pan, hanging from a knife-edge which rests on the rod in the place provided for it immediately underneath the scale. The scale is read by a small low-power horizontal microscope, furnished with

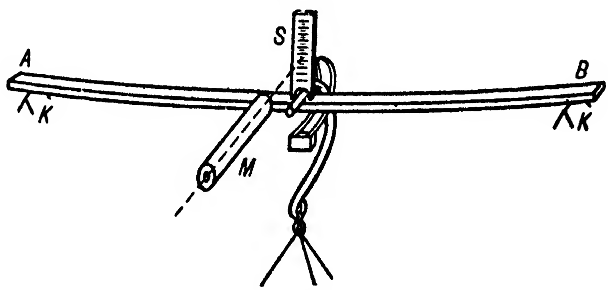


Fig. 68e.

cross-wires, fixed in a suitable position. To perform the experiment, the rod is placed symmetrically on the knife-edges, and the scale and the knife-edge carrying the scale-pan are placed in position. Readings of the scale are taken as the load in the pan is increased by equal increments up to a suitable load (not sufficient to bend the rod permanently), and as the load is decreased by the same steps to zero. The readings should agree very nearly. The distance between the supporting knife-edges is then measured with a mm. scale. It is the effective length of the rod, *i.e.* the l of the formula. The breadth and depth of the rod may be measured by vernier calipers. The experiment may be

repeated with a different distance between the supporting knife-edges, and with the bar standing edgewise instead of flatwise. The observed numbers will show that the depression of the centre of the rod is $\propto$ load, $\propto l^3$, and inversely $\propto bd^3$. The value of Y can be obtained by substitution in the formula of Art. 135c, Case II.

136e. Searle's Method for the Determination of the Young's Modulus and the Simple Rigidity of a Substance.—The two ends of a wire JS (Fig. 68*f*) made of the substance under test are rigidly fastened into two holes bored through the middle of two exactly equal metal bars, AB, CD, of square or circular cross-section. Metal hooks are fixed to the middle of these bars, and the system is suspended by means of equal pieces of cotton-thread attached to the hooks.

When suspended, AB and CD rest parallel. If now B and C are brought closer together and released, the bars vibrate approximately about their mid points and the wire vibrates transversely, bending from an approximately circular curve on one side of the straight line joining its ends to a similar curve on the other side.

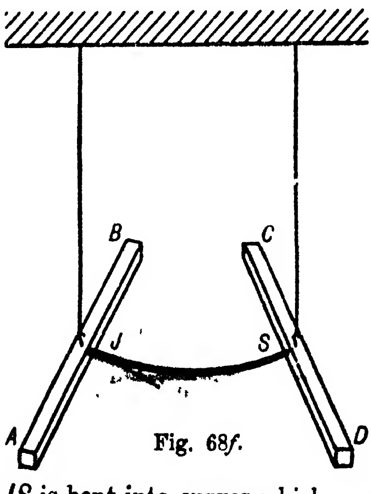


Fig. 68*f*.

Let us assume that the mass of the wire is negligible compared to the masses of the beams, that J and S remain fixed points, and that the wire JS is bent into curves which are arcs of circles.

At any given instant let θ be the angle through which each of the bars has been deflected from its position of rest; θ is also the angle between the tangents at the middle and ends of the wire.

Let R = radius of curvature of the wire in this position, then $R\theta = \frac{l}{2}$ where l is the length of the wire.

The bending moment at the ends of the wire due to either bar is, by Art. 135c, equal to $\frac{YI}{R}$ where Y = *Young's Modulus* of material of wire, and I = Moment of inertia of cross section of wire about its diameter

$$= \pi a^2 \times \tfrac{1}{4}a^2 \ (\S\ 76c) = \tfrac{1}{4}\pi a^4.$$

Substituting for R we get bending moment $= \frac{2YI\theta}{l}$.

This is also the value of the moment of the couple exerted by the wire on each bar.

Let K be the moment of inertia of each bar about a line through its centre of mass perpendicular to its length. Then the equation of motion of either of the bars is—

$$K\frac{d^2\theta}{dt^2} = -\frac{2YI\theta}{l};$$

or,

$$K\frac{d^2\theta}{dt^2} + \frac{2YI}{l}\theta = 0.$$

Let T_1 be the period of vibration of the system; then, by Art. 85, $T_1 = 2\pi\sqrt{\frac{Kl}{2YI}}$. .... (1)

Now let the cotton threads be removed and AB clamped horizontally, so that JS hangs vertically supporting CD. Set CD in vibration in a horizontal plane; by this means the wire will be set in torsional oscillation. Let T_2 be the period of vibration of CD; then, by Art. 123,

$$T_2 = 2\pi\sqrt{\frac{2Kl}{n\pi a^4}} \ \ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

where a = radius of cross section of the wire.

We can use the same K here as in (1) because the bars are square or circular.

Substituting $\frac{\pi a^4}{4}$ for I in equation (1) we get

$$T_1 = 2\pi\sqrt{\frac{2Kl}{Y\pi a^4}}; \text{ or, } Y = \frac{8\pi Kl}{T_1^2 a^4}. \ \ldots\ldots \ (3)$$

From equation (2) we get (as in Art. 123)

$$n = \frac{8\pi K l}{T_2^2 a^4} \quad \ldots\ldots\ldots\ldots \quad (4)$$

All the quantities on the right-hand side of equations (3) and (4) are determinate, hence we can find Y and n.

In performing the first part of the experiment it is well to set the system vibrating symmetrically. To do this the ends of the rods are brought together and a loop of cotton is thrown over them. To start the experiment, the cotton is burnt through. The timing of both parts of the experiments should be done by a chronometer or stop watch.

EXAMPLES IX.

1. Two balls of mass 20 gms., 60 gms. hang side by side suspended by strings 40 cm. long. The lighter one is drawn aside through a horizontal distance of 20 cm. and is set free. After the impact the large ball moves forward a horizontal distance 7·6 cm. Find e, and the velocity of the small ball after impact.

Call the balls A, B; call the angle through which the string is drawn θ. Then

$$\operatorname{Sin}\theta = \frac{20}{40}. \qquad \therefore \theta = 30°.$$

$\therefore$ Vertical height through which A is raised $= 40\,(1 - \cos 30°)$.
$\therefore$ velocity of A just before impact

$$= \sqrt{2gh} = \sqrt{2 \times 981 \times 40\,(1 - \cos 30)}$$
$$= 102 \text{ cm. sec.}^{-1}.$$

Similarly velocity of B just after impact $= 38$ cm. sec.$^{-1}$.
By momentum

$$20 \times 102 = 20x + 38 \times 60$$

if x is the velocity of A after impact,

$$\therefore x = -12.$$

Hence A *rebounds* with a velocity of 12 cm. sec.$^{-1}$.
To find e:

$$-e\,(102 - 0) = (-12 - 38)$$
$$\therefore e = \frac{50}{102} = \cdot 5.$$

2. A ball falls through a height of 50 cm. on a piece of plate glass. Find the change in velocity and its loss in kinetic energy.

($e = \cdot 7$, mass $= 2$ grm.)

3. Define simple rigidity; and explain carefully how the rigidity and the compressibility of a solid are connected with the elasticity to extension in a bar of the solid.

4. A lump of quartz which has been fused is suspended from a quartz fibre and allowed to oscillate under the influence of the torsion of the fibre. If the coefficient of the expansion of the material is 0·0000007 and the temperature coefficient of its rigidity is + 0·00013, how many seconds per day, or what fraction of a second per day, would a change of temperature of 1° C. make?

5. A spiral spring is stretched out to the 20·5 cm. graduation on a metre rule when loaded with a 10 grm. load, and to 23·4 cm. graduation by a 12 grm. load. If the spring be used to support a lump of metal (1) in air, (2) in water, the readings are 24·0 and 21·5. Find the mass and density of the metal.

Extension produced by 2 grm. $= 23{\cdot}4 - 20{\cdot}5 = 2{\cdot}9$ cm.

" " " metal in air $= 24{\cdot}0 - 20{\cdot}5 = 3{\cdot}5$ cm.

$\therefore$ Weight of metal $= 10 + \frac{3{\cdot}5}{2{\cdot}9} \times 2 = 12{\cdot}4$ grm.

Extension produced by metal in water

$= 21{\cdot}5 - 20{\cdot}5 = 1$ cm.

Weight in water $= 10 + \frac{1}{2{\cdot}9} \times 2 = 10{\cdot}7$ grm.

$\therefore$ Loss of weight = weight of water displaced $= 12{\cdot}4 - 10{\cdot}7$ grm.

$= 1{\cdot}7$ grm.

$\therefore$ Volume of metal $= 1{\cdot}7$ c.c.

$\therefore$ Density $= \frac{12{\cdot}4}{1{\cdot}7}$ grm. per c.c.

Hence density $= 7{\cdot}3$ grm. per c.c.

mass $= 12{\cdot}4$ grm.

6. A metal disc 10 cm. radius, mass 1 kg., is suspended in a horizontal plane by a vertical wire attached to the centre. If the diameter of the wire is 1 mm. and its length 1·5 m., and the period of torsional vibration of the disc is 5 seconds, find the rigidity of the wire.

7. Taking Young's modulus for steel as 2000 tonnes per sq. cm., find the load required to stretch a wire 10 m. long and 1 mm. in diameter through a distance of 5 cm.

8. A spiral spring carrying a load of 500 grm. vibrates three times in two seconds. What is the extension which the load produces? ($g = 980$ cm.)

9. A brass wire, 1 sq. mm. in section, 5 metres long, is stretched through a distance of ·5 cm. Find the average pull required to produce this elongation and the total work done. (Young's modulus = 800 tonnes per sq. cm.)

10. A watch has a non-compensated brass balance wheel and a steel balance spring. How many seconds will it lose per day for 1° rise of temperature if the coefficient of linear expansion of brass is $18 \div 10^6$, and if Young's modulus for steel at t° C. is

$$2 \times 10^{12}\left(1 - \frac{2t}{10^6}\right)?$$

11. Give some explanation of the fact that a body subjected to a sudden increase of pressure will rise or fall in temperature according as it expands or contracts on heating.

Describe an example of the phenomenon.

12. A mass of 1 kilogramme is suspended by an india-rubber cord, which then has a length of 1 metre and a diameter of 5 millimetres. The time of vertical vibration is found to be 1·5 seconds. Find the modulus.

13. If one body impinges on another which is at rest, what are the relations between (a) the momenta, (b) the energies of the system before and after impact?

In order to drive in a pile of 12 cwt. mass, the mean resistance to penetration being 2 tons weight, an inelastic mass of 4 cwt. is allowed to fall from a height of 10 feet upon it. Calculate the distance through which the pile will be driven at each blow.

14. Define the coefficient of rigidity of a substance, and explain how to determine it experimentally, proving any formulae used.

15. State the various laws which express the resistance of solids (1) to linear extension, (2) to linear compression, (3) to torsion.

16. A vertical rod, 15 ft. long, 6 sq. in. sectional area, carries a weight of 30 tons. It is stretched ¼ in. by the load. Find the numerical values of the stress and the strain.

17. Show that the dimensions of the equations in Articles 134, 135 are correct.

18. Maxwell's vibration needle consisted of a hollow tube suspended horizontally by a torsion wire. This tube could be filled by four equal cylinders each of length l, two of which were solid, and two hollow. If t_1 was the time of vibration when the two hollow cylinders were in the middle and the solid cylinders at the ends, and t_2 the time when the positions were interchanged, find an expression for the couple required to twist the wire through unit angle in terms of t_1, t_2, and the masses and lengths of the cylinders.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER X.

GRAVITY.

136. Galileo's Experiment.—Aristotle taught that bodies fall at rates depending on their weights: that the heavier a body the faster it should fall. This doctrine passed undisputed till the time of Galileo (1590), who asserted that all bodies fall at the same rate, unless they are so light as to be impeded by the air resistance. To prove his statements he dropped two different weights from the top of the leaning tower of Pisa. These started simultaneously and reached the ground simultaneously. The guinea and feather experiment shows the same thing. Drop the two together and the feather floats more slowly down through the air, but in a vacuum they both fall together. The reason that the acceleration of a falling body is constant is that the weight of a body is proportional to its mass. The weight of a two pound mass is twice the weight of a one pound mass. Hence the force causing a two pound mass to fall is twice the force causing a one pound mass to fall: but the mass on which it acts is also twice as great, so that the acceleration is the same in the two cases. This experiment of Galileo's is a proof that weight is proportional to mass. A variation of it is to show that the time of swing of a pendulum bob is the same whether the bob is hollow or filled with material of any density. This method was originally used by Newton. Galileo further illustrated his point by rolling bodies down inclined planes, showing that the acceleration, though dependent on the dimensions, &c., of the rolling bodies, was independent of their masses.

137. Methods of Finding g.—The acceleration of a falling body is usually denoted by g. Its value is different in different parts of the earth, being greatest near the poles. To determine it is not nearly such a simple matter as to show its constancy at any particular place. A very rough estimate of it may be found by dropping a leaden bullet through a height of sixteen feet. If the bullet be let go at one tick of a pendulum beating half-seconds, it will reach the ground on the next tick but one. The method is obviously not susceptible of any accuracy. The inaccuracy is not decreased by taking a larger fall, so that other methods have been devised. The pendulum gives us the best measurements, but other methods give reasonable results.

138. Atwood s Machine.—This machine is shown by the diagram, Figure 72. The two equal masses P, Q are connected together by a light silk thread passing over a pulley-wheel. To avoid friction as much as possible the axle of the pulley is supported on friction wheels. These are shown in Fig. 70. A rider R (Fig. 71) may be placed on the mass Q. This causes Q to descend and pulls up P.

Fig. 69.

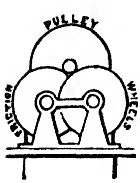


Fig. 70.

Fig. 71.

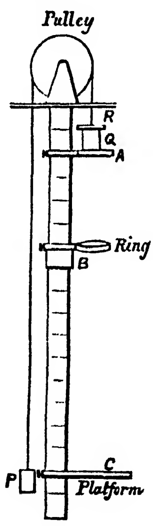


Fig. 72.

The motive force is the weight of R; the mass moved is the sum of the masses of P, Q, R; to this sum we must add a constant w, which represents the inertia of the wheel system. Denote the mass of P (or Q) and R by p and r. If f is the acceleration, then

$$rg = (2p + r + w)f,$$

i.e.

$$2p + r + w = \frac{rg}{f}.$$

Now repeat the experiment, using different masses p', but the same rider. We get

$$2p' + r + w = \frac{rg}{f'},$$

whence
$$2(p' - p) = rg\left(\frac{1}{f'} - \frac{1}{f}\right),$$

a result which does not involve the constant w.

Now the accelerations f, f' can be measured, for the motion is made slower than that of a body falling freely.

For convenience of starting and stopping the system two platforms A, C are arranged, A near the top, C towards the bottom. A can be released and then falls flat against the standard. B is a ring, adjustable in height, through which Q can pass freely, but which catches the rider R and removes it.

In using the machine, the mass Q with the rider resting on it is supported on the platform A (Fig. 72). The platform is allowed to fall, and the system then moves with constant acceleration till the rider is removed by the ring B, when the velocity becomes constant.

The details of the experiment can be varied considerably.

Perhaps the simplest way is to take the time of Q in falling between the ring and the lower platform. This may be done by means of a stop watch, water clock, or by so arranging the position of ring and platforms that the system starts at one tick of the seconds pendulum, reaches the ring at another tick, and strikes the bottom platform at a third. Suppose the distances between A and B and between B and C are a and c respectively, and that the time from A to B is t seconds, and the time from B to C is T seconds. Then, assuming that the acceleration (f) is uniform, we have

$$a = \tfrac{1}{2}ft^2,$$

and if v is the velocity on reaching the ring,

$$v^2 = 2fa.$$

The time taken to travel the distance c is $\frac{c}{v} = T$.

$$\therefore \quad \frac{c^2}{T^2} = v^2 = 2fa,$$

whence
$$\frac{c^2}{2aT^2} = f.$$

139. Notes.—(1) The *inertia of the pulley* wheel may be calculated. If its moment of inertia is Mk^2 and its radius b, when the mass Q has descended a distance s the work done by gravity is rgs. If v is the velocity of the system, its kinetic energy is

$$\tfrac{1}{2}(2p + r)v^2 + \tfrac{1}{2}Mk^2\frac{v^2}{b^2}.$$

This must be equal to the work done;

$$\therefore \quad v^2 = 2\frac{rgs}{\left(2p + r + \dfrac{Mk^2}{b^2}\right)}.$$

Hence the motion is the same as if the wheel were massless and the mass $2p + r$ increased by a mass $M\dfrac{k^2}{b^2}$, *i.e.* $w = M\dfrac{k^2}{b^2}$.

(2) The *friction* has been neglected. To eliminate this, find by experiment what mass placed on Q will, when the system (without rider) is set in motion, just keep it going. Add this mass to Q permanently.

The effect of friction may be eliminated by the following series of experiments. Call the retardation produced by friction a.

(*a*) Use a load $2p$ and rider of mass r_1

$$(2p + r_1 + w)(f_1 + a) = r_1 g.$$

(*b*) Use the same load $2p$ and a rider of mass r_2.

The frictional retardation will still be a, because the load resting on the axle is practically unaltered by substituting one small rider for another; hence

$$(2p + r_2 + w)(f_2 + a) = r_2 g.$$

Eliminate a from these equations: we get

$$f_1 - f_2 = g\left\{\frac{r_1}{2p + r_1 + w} - \frac{r_2}{2p + r_2 + w}\right\} \quad \text{...........(1)}$$

(*c*) and (*d*) are repetitions of (*a*) and (*b*), but with masses $2p'$ and riders r_3, r_4: we shall get

$$f_3 - f_4 = g\left\{\frac{r_3}{2p' + r_3 + w} - \frac{r_4}{2p' + r_4 + w}\right\} \quad \text{...........(2)}$$

From equations (1) and (2) we can eliminate w and so find g. The weights of the riders are always small compared with those of P and Q, so that we may write (1) in the form

$$f_1 - f_2 = g\left(\frac{r_1 - r_2}{2p + w}\right),$$

i.e.

$$2p + w = g\frac{r_1 - r_2}{f_1 - f_2}.$$

Similarly from (2) we get

$$2p' + w = g\frac{r_3 - r_4}{f_3 - f_4},$$

whence

$$g = \frac{2(p - p')}{\frac{r_1 - r_2}{f_1 - f_2} - \frac{r_3 - r_4}{f_3 - f_4}}.$$

(3) The *weight of the string* first retards the motion, but accelerates it when more than half of the string has passed over to the side on which Q is. Its effect is generally negligible compared with other errors.

Atwood's machine is an early attempt to find g, and is not without interest. The value of the method is however very small.

140. The Falling Plate.—This is a direct method of finding g. A smoked plate of glass is allowed to fall freely with its plane vertical. As it falls its surface is touched by a light spike on one of the prongs of a vibrating tuning-fork. The spike traces out a wavy line on the smoked surface (Fig. 73). The length of the wave is small at first and gradually increases. The period of the wave is constant and equal to that of the fork. This can be determined accurately. The distance travelled by the plate in an exactly measured interval of time can therefore be read off from the tracing. Two such measurements suffice to determine g.

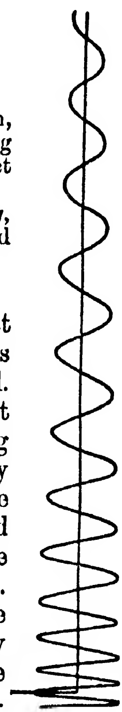


Fig. 73.

Suppose the frequency of the fork is n; that the distance travelled in a undulations is s_1, and the

distance in the next a undulations is s_2. Then, if u was the velocity at the beginning of the first a undulations,

$$s_1 = u\frac{a}{n} + \tfrac{1}{2}g\left(\frac{a}{n}\right)^2$$

$$s_1 + s_2 = u\left(\frac{2a}{n}\right) + \tfrac{1}{2}g\left(\frac{2a}{n}\right)^2,$$

$$\therefore\ s_1 - s_2 = \tfrac{1}{2}g\left(\frac{2a^2}{n^2} - \frac{4a^2}{n^2}\right),$$

i.e.

$$g = \frac{n^2}{a^2}(s_2 - s_1).$$

141. The Pendulum.—None of the methods described is of any great practical use. The only methods used for finding g are pendulum methods. The theory of the pendulum has been given in Chapter VII., and it is not necessary to discuss it further.

142. Borda's Pendulum.—This was an approximation to a simple pendulum. It consisted of a heavy platinum bob, spherical in shape, supported by a long, fine wire. The axis of suspension was a knife-edge.

Suppose a was the radius of the bob, l the distance from the knife-edge to the centre of mass.

The moment of inertia of the bob $= M\dfrac{2r^2}{5}$. (Art. 76)

The mass of the wire was small, so that its moment of inertia was neglected

The period was given by the relation

$$t = 2\pi\sqrt{\frac{l^2 + \frac{2r^2}{5}}{lg}}. \qquad \text{(Art. 92.)}$$

Borda used this pendulum to find the length of the seconds pendulum at Paris.

Another method of finding g is by timing the oscillations of a *uniform* brass rod about one end. If its length be l,

then

$$t = 2\pi\sqrt{\frac{2}{3}\frac{l}{g}}.$$

143. Reversible Pendulums.—One of the greatest advantages of Kater's pendulum (Art. 94) is that the only measurement in which great accuracy is required is that of the distance between the knife-edges. Knife-edges are definite and at places easily got at. The exact position of the centre of gravity of a pendulum cannot be readily found, and hence the use of the formula

$$t = 2\pi\sqrt{\frac{l^2 + k^2}{lg}}$$

involves the risk of considerable error. Of course l, the distance from axis of suspension to centre of gravity, can be calculated in certain cases where bob and rod are of definite shape, but want of uniformity in the material renders the method untrustworthy. In Kater's pendulum the sliding weight is moved till the times about the two axes are exactly equal. A somewhat easier device can be used which does not require such exactness. Suppose a pendulum is built after Kater's pattern, but without the sliding weight, the knife-edge being so placed that the times about the two knife-edges are very nearly, though not quite, equal. Let the times be t_1, t_2, and the distance of the centre of gravity from the edges l_1, l_2. Then if k is the radius of gyration about the centre of gravity we have

$$t_1 = 2\pi\sqrt{\frac{l_1^2 + k^2}{gl_1}}, \qquad t_2 = 2\pi\sqrt{\frac{l_2^2 + k^2}{gl_2}},$$

$$\therefore \frac{g}{4\pi^2}t_1^2 l_1 = l_1^2 + k^2, \qquad \frac{g}{4\pi^2}t_2^2 l_2 = l_2^2 + k^2,$$

$$\therefore \frac{g}{4\pi^2}\{l_1 t_1^2 - l_2 t_2^2\} = l_1^2 - l_2^2.$$

Let $$l_1 = l + x, \quad l_2 = l - x;$$

then $$\frac{g}{4\pi^2}\{(l + x)t_1^2 - (l - x)t_2^2\} = 4lx,$$

$$\therefore \frac{t_1^2 + t_2^2}{2l} + \frac{t_1^2 - t_2^2}{2x} = \frac{8\pi^2}{g}.$$

If $\frac{1}{2}(t_1 + t_2) = t$, and $\frac{1}{2}(t_1 - t_2) = \tau$,

$$\frac{t^2 + \tau^2}{l} + \frac{2t\tau}{x} = \frac{8\pi^2}{g}.$$

Here τ^2 can always be neglected in practice, and τ is so small that x need not be found accurately.

144. **Repsold's Pendulum** is built on this principle. It consists of a rod in which are fixed the two knife-edges, and on each end is placed a cylindrical bob, one of which is solid, the other hollow. Externally, however, they appear alike, so that the whole pendulum is symmetrical in form. The knife-edges are equidistant from the centre of symmetry, but at different distances from the centre of gravity.

145. **Effects of the Air.**—The period of a pendulum is affected by the air. In the first place the air buoys up the pendulum. To correct for this we must reckon that the weight of the pendulum is diminished by an amount equal to that of the air displaced.

Du Buat discovered that a pendulum bob carried with it a portion of the surrounding air. This makes a difference in the time of swing, and must therefore be reckoned with in dealing with ordinary pendulums. In a Repsold pendulum, however, no correction need be made if the observations of the times about the two axes are made at times when the density of the air is the same: for in this case the pendulum is symmetrical, so that the mass of air carried is the same in the two observations.

The resistance of the air tends to increase the period, but this effect is not important.

An experiment, due to Sir James South, clearly indicates that air is carried along with the pendulum. He attached to the spherical bob a piece of gold leaf which stuck out perpendicular to the surface. As the pendulum swung the leaf was not deflected, but remained in the same position, showing that the pendulum dragged the layer of air along with it.

146. **Errors.**—(1) *Temperature.* Several errors must be avoided or eliminated. In the first place, if the temperature changes during an experiment the length of the pendulum alters, thus affecting the period. To avoid this mistake care must be taken to keep the temperature constant while the rod is swinging and being measured.

The air effects are also dependent on temperature.

The scale used to measure the distance between the knife-edges must be corrected for temperature.

(2) *Vibration of the support.* If the support on which the knife-edges rest is not rigidly fixed it will vibrate in harmony with the pendulum, and the period of the pendulum will be dependent to some extent on the natural period of the support. The effect of this yielding of the support is virtually to make the centre of suspension either above or below the knife-edge.

In the case of an ordinary watch the balance wheel takes the place of the pendulum; the wheel is supported by the case. If, when the case is fixed, the watch goes correctly, it will not go correctly if hung from a nail. Lord Kelvin (*Popular Lectures and Addresses*) suggests some experiments which may be tried on a watch. Support it with the face horizontal by three strings attached to the frame. Each of these strings may be a yard long. The other ends of the strings are fastened to the lower side of a horizontal shelf and form an equilateral triangle. If this triangle is large, the watch is practically fixed and so goes at its normal rate. If the ends of the strings are closer together and form a small triangle, the reaction of the balance wheel on the rest of the watch sets up a motion of the whole, and thereby affects the rate of the watch.

(3) A correction must be made for the *curvature of the knife-edges.*

147. Method of Coincidences.—This method of comparing the times of two pendulums is applicable if the period of one is nearly equal to, or nearly a submultiple of, the period of the other. The pendulums, which we will call A and B, are placed one (A) in front of the other (B), and are arranged to swing in parallel planes. They are conveniently observed by a telescope. On the further pendulum (B) is a conspicuous mark, and the field of the telescope is confined in such a way that this mark can only be seen when B is in its equilibrium position. When A is also in its equilibrium position, it hides the mark.

Suppose the pendulum A has a period slightly less than that of B, so that A swings, say, 301 times while B swings 300 times. Imagine the two were started off together from the equilibrium position. On returning to this position after the first swing, A will be slightly in advance of B, so that the mark on B can be seen through the telescope as it passes; at the next swing A will have gained still more. A will

continue to gain on B till after 301 swings it will have just caught up B, which has only swung 300 times. At this passage the mark on B will be hidden by A. Hence the periods are compared. (See Example 2, p. 149.)

In finding the length of a seconds pendulum, the pendulum is placed in front of that of a clock beating seconds. The two are set going together, and the interval between two consecutive coincidences is noted on the clock.

148. Methods of Comparison of the Values of g at different Places.—The value of g varies with the locality: in general g is less at the equator than it is near the poles. This variation is due partly to the shape of the earth, partly to the fact that the earth is rotating. In order to find g at any place (X, say) one of two methods is usually employed. The first is by a determination of the exact length of the seconds pendulum at the place X; the other is to take to X a pendulum the period of which is accurately known at some place where g is also accurately known—say at Kew—and to compare its period at X with its period at Kew.

The second method is perhaps the better, though it requires some care to eliminate errors due to change (owing to temperature or other causes) in the length of the pendulum. For this purpose a pendulum beating half-seconds is convenient, for it is only one quarter of the length of a seconds pendulum and therefore much less cumbrous. For methods of comparison there is, of course, no need to know what the actual length of the pendulum is, so that an ordinary pendulum clock could be employed.

149. Gravitation.—All bodies fall to the earth with the same acceleration: from this it follows that the weight of a body is proportional to its mass, or that the attraction exerted by the earth on any body is proportional to the mass of the body. Newton extended this simple law, and gave in finished form the doctrine of universal gravitation. He was led to discover this law by his work in astronomy; before explaining his reasoning it will be well for us to review briefly the position of astronomy before his time.

Copernicus (born 1473) was the first to prove that the motion of the heavens was apparent only: that the stars were really at rest, that the earth moved round on its axis once a day and journeyed round the sun once a year. He pointed out, in fact, that the centre of our world was not the earth, but the sun, and that the earth was merely a planet like Jupiter or Mars. He believed the Ptolemaic doctrine that circles described round fixed or moving centres were the only paths that planets could have.

150. Kepler (1571-1630) tried to extend the work of Copernicus and to get more information about the Ptolemaic circles. He was aided in his attempt by the accurate observations which Tycho Brahé had previously made. He finally found that the motions of the planets were not composed of circular motions at all, but that each planet revolved round the sun in an ellipse.

Kepler's Laws.—Three laws are known by his name:

I. The path of a planet is an ellipse of which the sun is one focus.

II. If a line be drawn from the sun to a planet, then as the planet moves round the sun this line sweeps out equal areas in equal times.

III. The square of a planet's year (*i.e.* its time of revolution round the sun) is proportional to the cube of its distance from the sun.

These relations are not quite true. The deviations, however, are very small; and are due to the fact that the planets disturb one another in their orbits.

151. Isaac Newton (1642-1727) was acquainted with the results of Kepler's work. He also knew the three Laws of Motion, discovered by Galileo and clearly enunciated by himself:

(1) Every body continues in its state of rest or uniform motion unless acted on by a force.

(2) Change of momentum in a body is proportional to the force producing it and is in the direction of the force.

(3) To every action (or force) there is an equal reaction in the opposite direction.

These were the foundations on which the theory of universal gravitation was built.

152. Newton's Deductions from Kepler's Laws.—From the second of Kepler's laws Newton showed that it followed as a necessary consequence that the only force acting on a planet was a central force towards the sun, *i.e.* that the force which retained a planet in its orbit was a force in the line joining sun and planet. This was a great step forward: it swept away the theory of vortices in which Descartes had enveloped the universe to account for the motion of the planets. No tangential force was necessary to account for the permanence of the motion. From the first law Newton proved that the magnitude of the central force between sun and planet must vary inversely as the square of the distance between the two.

153. The Motion of the Moon.—The moon travels round the earth once a month. Its path is nearly a circle. To retain it in its orbit a force is necessary. The question occurred to Newton, Was this force of the same nature as the force which makes an apple fall? If so, then the acceleration of a stone towards the earth must be 60^2 times the acceleration of the moon towards the earth, for the moon is 60 times as far from the earth's centre as we are. The moon travels round the earth once in 27·3 days. Hence its velocity is $\frac{2\pi.60R}{27{\cdot}3 \times 24}$ miles per hour, R being equal to the radius of the earth in miles. The acceleration of the moon towards the earth is therefore $\left(\frac{2\pi.60R}{27{\cdot}3 \times 24}\right)^2 \frac{1}{60R}$ miles per hour per hour.

This acceleration $= \frac{1}{60^2} \times 32$ ft. sec.$^{-2}$.

That is to say the acceleration due to gravity at the surface of the earth is 60^2 times as great as the acceleration of the moon towards the earth.

This is the problem which was worked out by Newton. At first, however, his results were not satisfactory, for he did not know the radius of the earth, the received value of it being too small by about 12 per cent. Later, however, he heard of a fresh determination of the size of the earth made by Picard in Paris. He made the necessary correction and found agreement.

154. Attraction due to a Sphere.—There was also another difficulty to be removed. Take the case of the earth and a body near its surface. It would not be right to assume without proof that the earth may be replaced by a single particle of equal mass situated at its centre. The proof was given by Newton. He showed that the attraction exerted by a sphere of uniform density or of a sphere made up of concentric shells each of uniform density was the same as if the whole mass were concentrated at the centre. The like result does not hold for figures of other shapes: thus the attraction exerted by a cube or an ellipsoid is different from that due to a single particle of equal mass placed at its centre.

A formal proof of this proposition was necessary before the law of the inverse square could be extended from the attraction between celestial bodies to the attraction between the earth and a body near its surface.

Newton also showed that a spherical shell of uniform density exerted no force on a particle placed within it. It follows from this that the attraction at a point inside a homogeneous sphere is directly proportional to the distance of the point from the centre.

155. The Law of Universal Gravitation.—This may be stated as follows: Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of their distance apart. Thus if two particles have masses m, m', and are separated by a distance d, they attract one another with a force proportional to $\frac{mm'}{d^2}$.

We may write this if we please,

$$F = \lambda \frac{mm'}{d^2}.$$

λ is a constant which is independent of the nature of the masses.

Newton's Law deals with particles. In questions of astronomy the heavenly bodies are so far apart that we may look upon them as mere points, or regard their masses as concentrated at their centres of gravity.

156. Variation in g due to Rotation of the Earth.—As the earth rotates there is a tendency for bodies on its surface to be flung off, and were the rotation sufficiently rapid the gravitational attraction would not suffice to prevent it. The tendency is greatest at the equator, where the velocity is greatest, and least near the poles.

For the present we may regard the earth as a sphere of radius (r) 4,000 miles: the gravitational pull on a body will on this assumption be the same at all points of the earth. Call this pull on a one pound mass γ poundals. Let ω be the angular velocity of the earth about its axis. Suppose the one pound mass is suspended by a spring at some place on the equator. The forces acting on the mass are the tension w of the spring and the pull γ of the earth. Together these give the mass an acceleration $r\omega^2$ towards the centre of the earth.

$$\begin{aligned} \gamma - w &= r\omega^2 \\ &= 4000 \times 1760 \times 3 \times \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2 \\ &= \cdot 112 \text{ poundals} \end{aligned}$$

$$\therefore\ w = \gamma - \cdot 112;$$

i.e. the apparent weight at the equator of a one pound mass is less than its true weight by about ·112 poundals, or about $\frac{1}{3}$ per cent.

Now consider a place P at latitude λ (Fig. 74).

The plumb line will no longer be vertical and point to the centre of the earth, but will be inclined to the true vertical at a small angle. Adopt the same notation as before.

Resolve along the vertical line PO.

$\gamma - w \cos\theta =$ acceleration of the one pound mass towards O

$= (r \cos\lambda\omega^2) \cos\lambda,$

for the acceleration of the mass

$= PC\omega^2 = r \cos\lambda . \omega^2$

and is directed towards C.

Hence $\gamma - w \cos\theta = r \cos^2\lambda\omega^2$.

Now the deflection of the plumb line from the vertical is very small, so that we may consider $\cos\theta$ to be equal to unity.

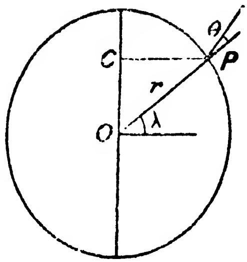


Fig. 74.

This gives $\gamma - w = r\omega^2 \cos^2\lambda$

or $w = \gamma - r\omega^2 \cos^2\lambda.$

157. The Bulge at the Equator.—The centrifugal force called into play by the earth's rotation is not the only cause which makes the value of g to change from place to place. Newton pointed out that unless the earth was extremely rigid its rotation must cause it to bulge out at the equator and to be flattened at the poles.

The equatorial radius of the earth is longer than the polar radius by about thirteen miles; hence bodies on the equator are further from the centre of mass of the earth than the bodies at the poles, and therefore the attraction exerted on a pound mass is less when the mass is at the equator than at the pole. The effect of the increased distance is not counterbalanced by the proximity of the bulge.

The relation $g = 980{\cdot}61 - {\cdot}025 \cos 2\lambda$ gives approximately the acceleration due to gravity in cm. per sec. per sec. at a place the latitude of which is λ. The constants in this formula have been found by direct experiment.

158. Variation in g due to Altitude.—The acceleration due to gravity at high altitudes is perceptibly less than at sea level.

Suppose a balloon at a height h, and suppose that the acceleration due to gravity on the earth's surface is g and in the balloon is g'. Then we get

$$\frac{g}{g'} = \frac{\frac{1}{r^2}}{\left(\frac{1}{r+h}\right)^2} = \frac{r^2 + 2rh + h^2}{r^2} = \left(1 + \frac{2h}{r}\right) \text{ approx.,}$$

whence
$$g' = g\left(1 - \frac{2h}{r}\right).$$

Combining this with the relation given in the preceding article we get the expression

$$\left(1 - \frac{2h}{r}\right)(980{\cdot}61 - {\cdot}025 \cos 2\lambda)$$

as a general expression for the acceleration due to gravity at a height h above sea level in a place of latitude λ.

The acceleration due to gravity on the top of a mountain is dependent on the effect produced by the mass of the mountain. The formula adopted by the Board of Trade in this case is

$$g = g_0\,(1 - {\cdot}00257 \cos 2\lambda)\left(1 - \frac{5}{4}\frac{h}{r}\right),$$

where g_0 is the value at a place of latitude 45°,
$g_0 = 980{\cdot}61$ cm. sec.$^{-2}$. (Everett's *Units and Constants.*)

159. Variations in g at Depths below the Surface of the Earth.—The attraction due to a uniform sphere at any point within it was shown by Newton to be directly proportional to the distance of the point from the centre. The earth is not a sphere of uniform density, so that we cannot expect the same result to hold for the attraction exerted by the earth on a body below the surface. As a matter of fact the weight of a body was shown by Sir George Airy to be greater at the bottom of a mine than at the surface. The result was obtained by pendulum experiments carried out in the Harton Colliery in Durham. This mine was some 1200 feet deep. His results have been confirmed by later experiments carried out in deeper mines. The reason that g at first increases with the depth is that the central portions of the earth are much denser than the crust.

160. Shape of the Earth.—The earth may be shown by methods of surveying to be very nearly an oblate spheroid. Any section of it through its axis of revolution is an ellipse. The length of a meridian circle drawn across the equator through the poles is about 40,008,000 metres. This differs by eight kilometres from the supposed length of the arc from which the definition of the metre was derived.

Arc from pole to equator = 10,002,000 metres.

The equatorial circumference = 40,097,000 metres.

The equatorial radius is 6,378,300 metres long, the polar radius is 6,356,500.

161. Gravitation Constant—In the law of gravitation, $F = \mathbf{G}.mm'/d^2$, the factor **G** is not easily determined. For most purposes its value is not important. Its determination brings with it a knowledge of the density of the earth (Δ). There are two ways in which it may be found:

(1) The first involves measuring the attraction which a portion of the earth—*e.g.* a mountain range—of known constitution exerts on a pendulum.

(2) The second method was first used by Cavendish. It consists in measuring by torsion balance or other means the attraction of two small masses for one another.

Tidal Method.—A third method of finding **G** is by measuring the attraction of the mass of water brought up by high tide into an estuary. This alters the direction of gravity at an observatory north or south of the estuary and so alters its latitude. Its great advantage over the mountain method is that the density of the water is known quite accurately: so is the thickness of the slab of water. The shape of the estuary is determined by careful survey.

162. The Mountain Method.—In 1735 an expedition went out to Peru to measure the length of a degree of longitude in that place, the object being to determine the figure and dimensions of the earth. Bouguer, who went with the expedition, attempted to find out how much Mount Chimborazo drew the plumb line out of the vertical. His attempt met with fair success, though not much reliance could be placed on the value of **G** or Δ obtained from his results. Practically the same method was adopted by Nevil Maskelyne some forty years later. He experimented at Mount Shiehallien.

163. The Shiehallien Experiment.—Suppose two places *A* and *B* are selected, *A* being due north of *B*. The arc *AB* subtends an angle to the centre of the earth which is approximately equal to $\frac{d}{6400}$, where *d* is the distance from *A* to *B* in kilometres, and the angle is measured in circular measure. This angle is generally equal to that between two plumb lines, one hung at *A*, the other at *B*; but if any large mountain mass intervene between *A* and *B*, then the two lines may be deflected to a slight extent. To measure the angle between the verticals it is sufficient to determine the difference in latitude between *A* and *B*. This may be done with great accuracy by determining the altitude of the pole star.

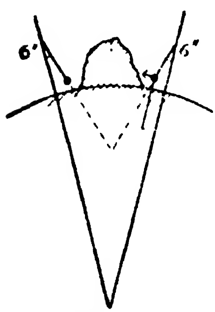


Fig. 75.

In the Shiehallien experiment it was found that the angle subtended at the centre of the earth by the arc joining the two selected places was about 50 seconds. The difference in latitude obtained by astronomical readings was about 12 seconds greater. This practically means that the mass of the mountain deflected a plumb line on each side of the mountain by about 6 seconds.

These readings afford a comparison between the attraction due to the mountain and that due to the earth. The mass of the mountain can be estimated; hence that of the earth.

The experiment is easily imagined, but difficult to carry out. It is very hard to estimate correctly the mass of the mountain. The experiments have been repeated at other places, but exact agreement is not obtained.

Mount Shiehallien is in the south of the Grampians in Perthshire. It was selected by Maskelyne because its geological constitution was fairly simple. It is a ridge of schist 3,500 feet high, running east and west.

164. Cavendish's Experiment.—Henry Cavendish was associated with Maskelyne in the Shiehallien experiment. He afterwards measured the density of the earth by measuring the attraction between two balls of lead. His apparatus is shown in Figs. 76, 77.

C, *D* are two heavy balls of lead (12 in. in diameter) suspended from a beam *PQ*. This beam could be made to rotate through any angle about the vertical axis *XX*.

Beneath this beam is a lighter rod *RS*, suspended by a torsion wire *T*. From the ends of this were hung two

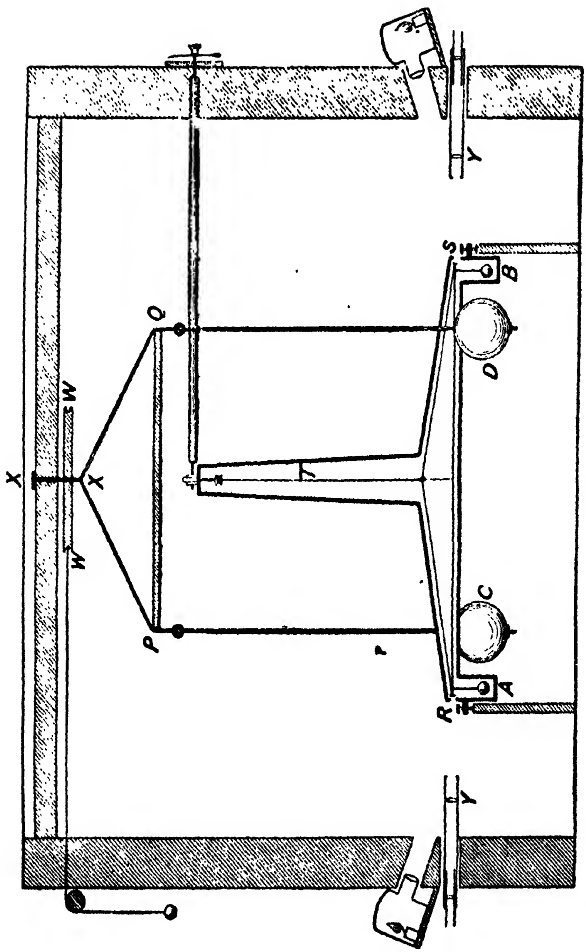


Fig. 76.

small balls of lead A, B (2 in. in diameter). The centres of the four balls $A\,B\,C\,D$ all lay in the same horizontal circle of radius about 3 feet.

A plan of the apparatus is shown in Fig. 77: $s\,s\,s\,s$ are screws supporting the case.

The method of experiment was as follows:—

The beam PQ carrying the heavy balls C, D was first turned round the vertical till its direction was at right angles to that of the light rod RS. The torsion wire was

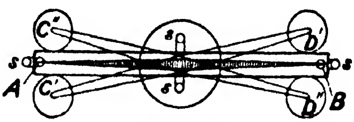


Fig. 77.

then without twist. The exact position of the line joining A, B was noted.

The beam was then swung round till the balls C, D moved into the position C', D' (Fig. 77). The attraction they exerted on the lighter balls twisted the torsion wire and brought the centres of A and B into positions A', B' (not shown in the figures). This new position was carefully observed.

The beam was next swung round the other way till the balls took up the positions C'', D''; A'', B''; and the twist of the wire again noted.

The theory of the torsion balance has been already given (Art. 122). The modulus of the wire was found by a previous experiment.

165. Calculation of Result.

Let

$M =$ mass of each of the large balls.

$m =$ mass of each of the small balls.

$d =$ distance of centres of A, C when the deflection of the rod was a.

$2l =$ length of rod RS.

Let the couple exerted by the wire when twisted through an angle θ be $\mu\theta$.

The force between the balls A and C

$$= G\frac{Mm}{d^2}.$$

The moment about the axis of the attractions exerted by one pair of balls on the other

$$= 2Gl\frac{Mm}{d^2}.$$

This was balanced by the couple (μa) exerted by the wire.

Hence $$\mu a = 2Gl\frac{Mm}{d^2},$$

i.e. $$G = \frac{\mu a d^2}{2lMm}.$$

The measurements to be taken in an experiment such as this are so refined that special caution must be taken to avoid absurd results. Cavendish kept his apparatus in a closed room which he did not enter while making his experiment. The readings were taken through telescopes Y, Y (Fig. 76). The beam PQ was turned by means of a string fastened to a wheel WW. The temperature had to be kept constant, otherwise draughts had more effect on the balls than the gravitational attraction.

To avoid electrostatic attractions from outside the apparatus was placed in a gilt covered glass case. This precaution also served to equalise the temperature.

The balls were not observed at rest: the method employed was to notice the extent of the swing. This is exactly the same principle as that used in weighing by oscillation (see Art. 37).

To deduce the density of the earth we equate the weight of a body to the force of attraction between it and the earth. Thus let m be the mass of a body, and Δ and r the density and radius of the earth. The weight of a body on the surface of the earth $= mg$. The force of attraction between it and the earth $= G\frac{\left(\frac{4}{3}\pi r^3\Delta\right)m}{r^2}$.

Putting $mg = \frac{4}{3}\pi\ Gr\Delta m$ we get $\Delta = \frac{3g}{4\pi Gr}$.

166. Boys's Experiments.—The Cavendish method has been employed by several other experimenters: notably by Reich, Baily, by Cornu and Baille, and by Boys. With the last only need we concern ourselves. Prof. Vernon Boys discovered the merits of fibres drawn from fused quartz. Such fibres can be obtained extremely thin, and

are highly elastic. He replaced the torsion wire of Cavendish by such a fibre, and was thus enabled to reduce the scale of the apparatus to a considerable extent without loss of accuracy.

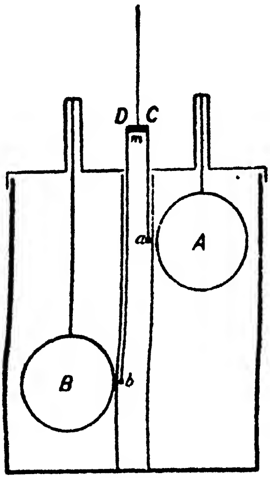


Fig. 78.

Fig. 78 is a diagram of one form of this apparatus.

The two balls a, b were hung by quartz fibres from the ends of the bar CD, which was a combination of beam and mirror. They were accurately made spheres.

The beam DC was so short (·9 in. long) that it was necessary to arrange a, A on one level, b, B on another. Otherwise the effects of A and B would have interfered.

The balls A, B were brought up into position by revolving the top of the case.

The balls a, b were of gold, about $\frac{1}{4}''$ diameter. A and B were of lead and weighed about $7\frac{1}{2}$ kilogrammes.

The results obtained varied from

$$\Delta = 5{\cdot}5159 \quad \text{to} \quad 5{\cdot}5306$$
$$G = 6{\cdot}6711 \times 10^{-8} \quad \text{to} \quad 6{\cdot}6533 \times 10^{-8}.$$

(*Phil. Trans.*, 1895.)

167. Values found for the Density of the Earth.

Experimenter.	Method.	Result.
Maskelyne	Mountain	$\Delta = 4{\cdot}7$.
Cavendish	Torsion balance	5·45.
Cornu and Baille	Torsion balance	5·56.
Poynting	Weighing	5·49.
Boys	Torsion of quartz fibre	5·53. ($G = 6{\cdot}66 \times 10^{-8}$).

EXAMPLES X.

1. In a Kater's pendulum the times about the two knife-edges are t, $t+\tau$, where τ is small: the knife-edges are distant l, l' from the centre of gravity. Prove that

$$l + l' = \frac{gt}{4\pi^2}\left\{t + \frac{2l'}{l'-l}\tau\right\}.$$

2. A simple pendulum supposed to beat seconds is observed against an accurate clock by the method of coincidences. A coincidence occurs at 5 h. 12 m. 6 s., and the next at 5 h. 18 m. 14 s. What is the true period of the pendulum?

The clock pendulum has vibrated 368 times. Hence the other has vibrated either 369 or 367 times so that its period is either

$\frac{368}{369}$ sec. or $\frac{368}{367}$ sec., *i.e.* ·997 sec. or 1·003 sec.

3. How do we know that the weight of a body varies in different parts of the world? Explain the causes of this variation.

4. Show how to find the attraction on a particle due to an infinitely long straight wire of line density ρ.

5. Find the attraction due to an infinite plane of surface density ρ.

6. State the law of the variation of the force of gravitation between two particles with the distance between them. Why is it legitimate in the case of falling bodies to assume the value of g to be constant?

7. The radius of the moon's orbit is 240,000 miles, and the period of revolution is 27 days; the diameter of the earth is 8,000 miles, and the value of gravity on the surface is 32 feet per second per second. Verify the statement that the gravitational force varies inversely as the square of the distance.

8. The mass of a railway train is 100 tons. What will be its weight when travelling (1) due East, (2) due West along the equator at 60 miles per hour?

9. Describe Atwood's machine, and devise numerical illustrations by its means in proof of the first law of motion, and also in proof of the relation between the space described and the time of description in the case of a body under the influence of a constant force.

10. Two equal balls are hung up side by side by threads 10 metres long. If the distance between the upper ends of the thread is 1 cm., find by how much the threads are pulled out of the vertical by the attraction of the balls.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER XI.

GASES.

168. General Characteristics of Gases.—It has already been pointed out in Chapter III. that it is difficult to draw any hard and fast line between the three divisions of matter. Though gases differ in many ways from liquids and solids, the differences are often merely in degree. Thus gases are generally light, but when near their point of liquefaction are heavier than light liquids. Their refractive indices are low, they are poor conductors of heat and—in their normal state—of electricity; they are generally slightly coloured or colourless, very transparent, and diathermanous. They mix together in all proportions: they have no rigidity, they are practically without cohesion, and exert little frictional resistance to motion. They have weight and are controlled by gravity, and yet are miscible among themselves in all proportions.

Their most distinguishing characteristic, however, is their compressibility and extensibility. When pressure is applied to solids or liquids changes in volume so produced are small, and if pressure be wholly removed by placing them under the receiver of an air pump no noticeable change takes place. A gas on the other hand, though it has a limit beyond which it cannot be compressed, yet with the removal of external pressure expands indefinitely, so that it always occupies the whole of the space at its disposal.

169. Robert Boyle appears to have been the first to study carefully the behaviour of gases under pressures.

His experiments on the "Spring of Air" are described in his *Defence of the Doctrine touching the Spring and Weight of Air.* This was published in the year 1662. Some of his previous work on gases had been criticised by Linus, a Dutch professor, who affirmed that air was too

feeble a substance to hold up the mercury column in a barometer, and maintained a funicular hypothesis asserting that the mercury was held up by invisible threads fastened to the glass of the tube. Boyle in his defence against Linus describes the experiments he made which led up to the discovery of the great law now known by his name.

He prepared two series of experiments for testing the effects of pressure on air: in the first the air was subjected to a greater pressure than that of the atmosphere, and in the second to less.

His apparatus is almost identical with what is now used in laboratories, so that it will be of interest to describe his experiments in his own words:—

"We took then a long Glass Tube, which by a dexterous hand and the help of Lamp was in such a manner crooked at the bottom, that the part turned up was almost parallel to the rest of the Tube, and the orifice of this shorter leg of the Siphon (if I may so call the whole Instrument) being Hermetically seal'd, the length of it was divided into Inches, (each of which was subdivided into eight parts) by a straight list of paper, which containing the subdivisions was carefully pasted all along it: then pouring in as much Quicksilver as served to fill the Arch or bended part of the Siphon, that the *Mercury* standing in a level might reach in the one leg to the bottom of the divided paper, and just to the same height or Horizontal line in the other; we took care by frequently inclining the Tube, so that the Air might freely pass from one leg into the other by the sides of the *Mercury*, (we took (I say) care) that the Air at last included in the shorter Cylinder should be of the same laxity with the rest of the Air about it. This done we began to pour Quicksilver into the longer leg of the Siphon, which by its weight pressing up that in the shorter leg, did by degrees streighten the included Air: and continuing this pouring in of Quicksilver till the Air in the shorter leg was by condensation reduced to take up but half the space it possess'd (I say *possess'd*, not *filled*) before; we cast our eyes upon the shorter leg of the Glass, on which was likewise pasted a list of Paper carefully divided into Inches and parts, and we observed, not without delight and satisfaction, that the Quicksilver in that longer part of the Tube was 29 Inches higher than the other. Now that this Observation does both very well agree with and confirm our *Hypothesis*, will be easily discerned by him that takes notice that we teach, and Monsieur *Paschall* and our *English* friends Experiments prove that the greater the weight is that leans upon the Air, the more forcible is its endeavour of Dilatation, and consequently its power of resistance, (as other Springs are stronger when bent by greater weights) and this being considered it will

appear to agree rarely-well with the *Hypothesis*, that as according to it the Air in that degree of density and correspondent measure of resistance to which the weight of the incumbent Atmosphere had brought it, was able to counterbalance and resist the pressure of a Mercurial Cylinder of about 29 Inches, as we are taught by the *Torricellian* Experiment; so here the same Air being brought to a degree of density about twice as great as that it had before, obtains a Spring twice as strong as formerly. As may appear by its being able to sustain or resist a Cylinder of 29 Inches in the longer Tube, together with the weight of the Atmospherical Cylinder, that leaned upon those 29 Inches of *Mercury*; and as we just now inferred from the *Torricellian* Experiment, was equivalent to them.

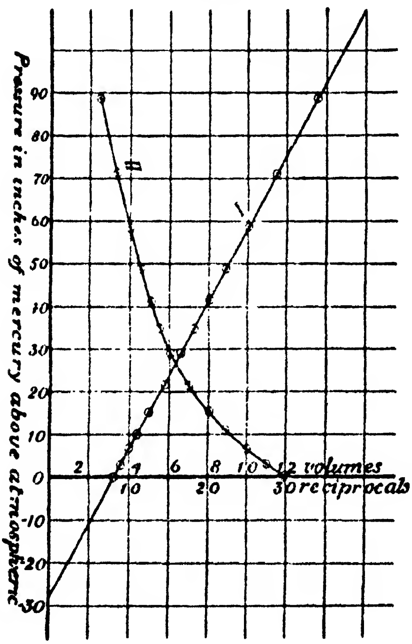


Fig. 79.

"We were hindered from prosecuting the tryal at that time by the casual breaking of the Tube. But because an accurate Experiment of this nature would be of great importance to the Doctrine of the Spring of the Air, and has not yet been made (that I know) by any man; and because also it is more uneasy to be made than one would think, in regard of the difficulty as well of procuring crooked Tubes fit for the purpose, as of making a just estimate of the true place of the Protuberant *Mercury's* surface; I suppose it will not be unwelcome to the Reader, to be informed that after some other tryals, one of which we made in a Tube whose longer leg was perpendicular, and the other, that contained the Air, parallel to the Horizon, we at last procured a Tube of the Figure exprest in the Scheme; which Tube, though of a pretty bigness, was so long, that the Cylinder whereof the shorter leg of it consisted admitted a list of Paper, which had before been divided into 12 Inches and their quarters, and the longer leg admitted another list of Paper of divers foot in length, and divided after the same manner: then Quicksilver being poured in to fill up the bended

part of the Glass, that the surface of it in either leg might rest in the same Horizontal line, as we lately taught, there was more and more Quicksilver poured in to the longer Tube; and notice being watchfully taken how far the *Mercury* was risen in that longer Tube, when it appeared to have ascended to any of the divisions in the shorter Tube, the several Observations that were thus successively made, and as they were made set down, afforded us the ensuing Table."

Instead of giving his actual figures we have depicted his results graphically in Fig. 79. In these the ordinates denote the pressure in inches of mercury, while the abscissae are proportional to the reciprocal of the volume in (i), and to the volumes in the rectangular hyperbola (ii).

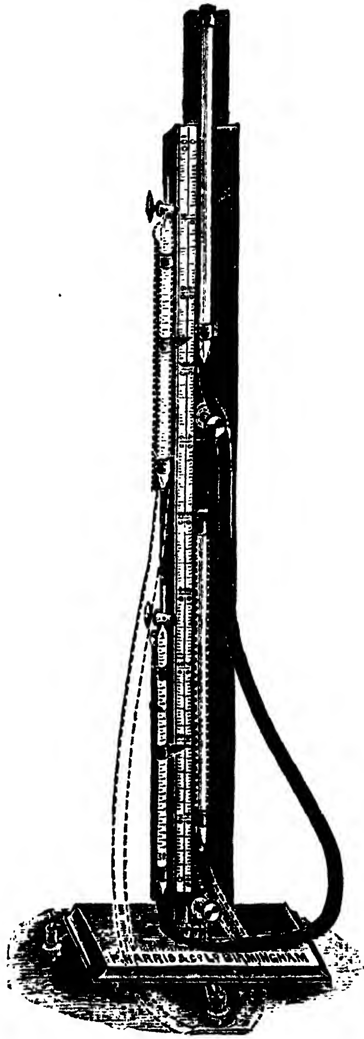


Fig. 80.

170. Boyle's Law was rediscovered at a later date by the French physicist Mariotte. The law is consequently often spoken of as Mariotte's Law. It may be stated as follows: The volume of any quantity of gas varies inversely as the pressure to which it is subjected.

Since the density of any mass is inversely proportional to its volume, it follows that the density of a gas is proportional to its pressure.

Figure 80 represents a convenient form of apparatus for experiments on Boyle's Law.

171. The Nature of a Gas: Kinetic Theory.—The mechanics of a gas and its nature may now be considered. Is a gas a continuous medium or does it consist of small separate particles? Gases mix with one another so freely that it is difficult to conceive the possibility of their being continuous media, for how could two continuous media occupy the same space at the same time? Could a continuous medium on reduction of pressure expand indefinitely? If we assume an atomic structure we have to account for the pressure exerted by a gas, for the fact that light gases will diffuse downwards into heavier, that a rise in temperature produces an increase of pressure or volume, that in the most perfect vacua we can produce the distribution of the residual gas appears to be uniform. The Kinetic Theory of gases explains all these phenomena, and has predicted other results which, though unexpected, are yet found to be in agreement with experiment. The conduction of electricity through gases, Avogadro's Law, the effects of viscosity are all consonant with the theory. The theory supposes that gases are made up of small particles in rapid motion. These fly about in all directions, continually encountering one another and so changing their speeds and directions. When in a confined space they impinge on the sides of the containing vessel and rebound again. The particles must be very small, their velocities great, and their number per cubic centimetre enormous. A gas then cannot be at rest; it is always in violent motion and its equilibrium is dynamic equilibrium, not static.

Another example of similar dynamic equilibrium may be found in the case of certain chemical reactions. Thus, when a piece of chalk

is heated in a closed vessel it suffers a partial decomposition. A definite quantity dependent on temperature and pressure breaks up into lime and carbon dioxide. Equilibrium when established is, however, only apparent; the chalk continues to break up, but some of the lime and carbon dioxide unite again to make up the loss, and the two processes exactly counterbalance.

The gas particles as they rebound from the sides of the vessel change their directions and their momenta. This bombardment of the walls tends to drive the walls outward and so exerts pressure on them. The pressure is in accordance with Newton's Second Law, measured by the rate of change of momentum. On this supposition it is easy to account for Boyle's Law. For assume that in a closed vessel there are n gas particles, and that these exert a pressure p on the sides. Now crush into the vessel n other similar particles endowed with the same motion as the first set. The effect of this is to double the density of the gas; also the number of impacts per second on the sides will be doubled, so that the pressure will rise from p to $2p$: hence the ratio of pressure to density is unaltered, and Boyle's Law is obeyed.

172. Thermal Effects.—It is well known that when a gas is allowed to expand its temperature falls, and that when compressed its temperature rises. To account for these effects let us consider how gas particles contained in a cylinder are affected by the motion of the piston. When the piston is at rest a particle whose velocity is normal to the piston rebounds after impact with the same speed. If, however, the piston is retreating the particle will return with a reduced velocity. The average velocity of the particles is thus diminished by increase in volume. Similar reasoning shows that the effect of compression must be increase of average velocity. It is therefore not unreasonable to suppose that the temperature of a gas is dependent on the velocity of its particles. The physical difference between a gas when hot and the same gas when cold is that when hot the particles are in more rapid motion and are possessed of more energy. This explanation would lead us without difficulty to forecast the effects of warming a gas: the particles move faster and so strike their enclosure harder, causing a rise in pressure. The reasonableness of these

assumptions is increased by a study of the nature of heat and the ways in which it is generated. Energy of all kinds generates heat: when energy of any motion seems to disappear a rise in temperature can nearly always be detected, so that it is abundantly possible that heat is a mode of motion—energy not of visible motion of a body as a whole, but of rapid motion and vibration of its constituent particles.

173. Elementary Mathematical Investigation.—Suppose a certain mass of gas is kept in a closed space. To make our conceptions definite, consider it shut up in a cube of one centimetre edge. Let the number of particles be n, their average velocity v, and the mass of each m. Now the particles fly about in all directions, encountering one another and the sides of the vessel, so that no single particle keeps its line of flight very long. Its velocity at any instant may, however, be resolved into three components parallel to the edges of the cube: the average of one of these components, taken over a sufficient interval of time, will be equal to the average of either of the other two, so that it would seem fair for the purposes of calculation to consider that the particles are not moving haphazard in all kinds of directions, but that they are divided into three equal sets with velocities parallel to three adjacent edges of the cube. At any instant we may assume that $\frac{1}{3}n$ particles are moving with average velocity v, parallel to a particular edge, and therefore perpendicular to two particular faces, A and A' say.

Any one of these particles would pass from face to face in time $\frac{1}{v}$, so that it would strike the face A $\frac{v}{2}$ times per second. By impact its velocity is on the average completely reversed, so that its momentum undergoes a change on each occasion of $2mv$, *i.e.* from mv to $-mv$. The total change of momentum per second is therefore

$$\tfrac{1}{3}n \,.\, 2mv \,.\, \frac{v}{2} = \tfrac{1}{3}nmv^2.$$

This measures the force exerted on the face A, for by Newton's Second Law of Motion force is measured by the

change of momentum produced per second. Force on unit area is the same as pressure, so that we get the relation

$$p = \tfrac{1}{3}nmv^2 \qquad\qquad (1)$$

Now nm is the mass of unit volume of the gas, *i.e.* the density. Call this ρ, and (1) becomes

$$= \tfrac{1}{3}\rho v^2 \qquad\qquad (2)$$

so that $p \propto \rho$ if v is constant. This expresses Boyle's Law: the pressure of a gas varies as its density if the temperature is constant.

The results 1, 2 are not quite accurate. The velocity, v, with which we have been dealing is the average velocity. For this we ought to substitute a velocity termed the mean value of the molecular speed or the velocity of mean square. This is defined as square root of the mean of the squares of the speeds of the molecules in the gas: if we denote it by U, we shall then have it defined by the equation

$$U^2 = \frac{1}{n}(v_1^2 + v_2^2 + \dots v_n^2),$$

where $v_1, v_2 \dots v_n$ are the actual speeds of the particles

The velocity U is greater than v, the average velocity

$$\left(\frac{v_1 + v_2 + \dots v_n}{n}\right).$$

The two are connected by the relation

$$v = \sqrt{\frac{8}{3\pi}}\, U = \cdot 9213U = \tfrac{12}{13}U \text{ (approx.)} \qquad\qquad (3)$$

Equations (1), (2) now become

$$p = \tfrac{1}{3}nmU^2 \qquad\qquad (4)$$

$$p = \tfrac{1}{3}\rho U^2 \qquad\qquad (5)$$

We may get these results in another way.

Suppose V is the velocity of a molecule: resolve it into its components, u, v, w, parallel to the sides of the vessel. Then

$$V^2 = u^2 + v^2 + w^2.$$

Deal with the other molecules in the same way. Then, as before, we can show that the pressures exerted on the sides are

$$\Sigma mu^2, \quad \Sigma mv^2, \quad \Sigma mw^2.$$

These are all equal, so that

$$\begin{aligned} p &= \Sigma mu^2 = \Sigma mv^2 = \Sigma mw^2 \\ &= \tfrac{1}{3}\Sigma m(u^2 + v^2 + w^2) \\ &= \tfrac{1}{3}\Sigma m V^2 \\ &= \tfrac{1}{3} nm U^2. \end{aligned}$$

174. Molecular Speed.—Equation (5) of Art. 173 enables us to calculate the mean speed of a molecule.

From it we obtain $U = \sqrt{\dfrac{3p}{\rho}}$ ....(6)

All the quantities occurring in this formula are measured on the C.G.S. absolute system. Taking the case of hydrogen at a temperature of 0° C. and a pressure of 760 mm. of mercury, we have

$$\rho = \cdot 00009 \text{ grm. per c.c.}$$

$$p = 76 \times 13{\cdot}6 \times 980 \text{ dynes per sq. cm.}$$

Substituting these values in (6) we get $U = 180000$ cm. per sec.

Hence the velocity of mean square in the case of a hydrogen molecule is 1800 metres per second. The mean velocity $= \frac{12}{13} \times 18000 = 170000$ cm. sec.$^{-1}$.

Lord Rayleigh gives the values

$$\left.\begin{aligned} U &= 183820 \\ v &= 169360 \end{aligned}\right\} \text{cm. per sec.}$$

The speeds of molecules are by (6) inversely proportional to the square roots of their densities, and may therefore be easily deduced from that of hydrogen. Thus, referred to hydrogen, the specific gravity of oxygen is 16, and that of carbonic acid is 44.

The molecular speeds (U) of these gases (at 0° C.) are therefore

$$\frac{1838}{\sqrt{16}} \text{ m. sec.}^{-1}, \text{ and } \frac{1838}{\sqrt{44}} \text{ m. sec.}^{-1},$$

or 460 and 280 metres per second respectively.

175. Temperature.—Equation (5) of Art. 173 may be written in the form

$$\frac{p}{\rho} \propto U^2 \text{ ....(7)}$$

Now from Charles' Law and Boyle's Law we know that

$$pv = RT,$$

where v is the volume of a gas and T its absolute temperature;

$$\therefore \frac{p}{\rho} \propto T \text{ ....(8)}$$

Combining (7) and (8) we get

$$T \propto U^2 \text{ ....(9)}$$

The kinetic theory therefore leads us to the conclusion that the absolute temperature of a gas is directly proportional to the square of the speed of its molecules.

176. Avogadro's Hypothesis.—Clausius and Maxwell have shown that if two gases are mixed together and are in a state of equilibrium, then the average kinetic energy of one set of molecules is equal to the average kinetic energy of the other set. Assuming this result, consider the case of a mixture of equal volumes of two gases, say oxygen and nitrogen, at the same temperature and pressure. Denote the pressure, volume, etc., of the second gas by dashed letters, then

$$\tfrac{1}{2}mU^2 = \tfrac{1}{2}m'U'^2 \text{ ....(10)}$$

Now as no change can be detected on mixing two gases which have no chemical action on one another, we assume the velocities to be unaltered. Hence (10) held when the gases were still separate, so that we have

$$\tfrac{1}{3}nmU^2 = p = p' = \tfrac{1}{3}n'm'U'^2 \text{ ....(11)}$$

Hence $$n = n',$$

i.e. equal volumes of gases contain the same number of molecules.

177. Assumptions.—Among other assumptions which have been made in the above investigations, the following are important:

(1) There is no loss of kinetic energy at an encounter.

This is legitimate, for there are thousands of encounters per second, so that if energy were lost at any measurable rate at an encounter the universe would lose its energy in a very short time.

It is not necessary to suppose that the molecules behave exactly like elastic spheres, and bounce back from the walls of the containing vessel with unaltered speed: as a matter of fact this is not the case. We do, however, assume that *on the average* the speed is just reversed.

(2) The molecules exert no sensible attraction or repulsion on one another except during encounters. (*v.* Art. 182).

(3) The time spent in encounters is enormously less than that spent in the free path between the encounters, and this free path is straight.

178. Despretz.—For a century and a half after the time of Boyle very little advance was made in the study of gases. In the year 1827 Despretz carried out experiments on different gases and compared their behaviour with that of air. Equal amounts of gases were admitted to barometer tubes placed side by side, the mercury level being initially the same in each. Pressure was then applied by putting them in a strong glass vessel filled with water and closed by a screw. As the screw was turned the pressure rose, and the gases were compressed, but not equally. Gases easily liquefied, *e.g.* carbonic acid, cyanogen, ammonia, were compressed more than air and hydrogen. Between air and hydrogen no difference could be detected till a pressure of some fifteen atmospheres was applied. At higher pressures hydrogen was slightly but distinctly less compressible than air.

From these experiments it was evident that all gases could not strictly follow Boyle's Law, but whether air did or not was left undetermined.

179. Regnault.—Somewhat later Regnault made careful experiments, taking great care to measure the volumes accurately. His results showed that for all the gases except hydrogen on which he experimented, the product pv instead of remaining constant decreased as the pressure rose. In hydrogen—"*gaz plus que parfait*"—however, pv increased. His conclusion therefore was that with the exception of hydrogen all gases were more compressible than would be a perfect gas that obeyed Boyle's Law.

180. Natterer attempted to liquefy the permanent gases. In doing this he employed much higher pressures than Regnault, and showed that the compressibility of nitrogen and air ceased to decrease as the pressure rose, but reached a minimum at about 50 atmospheres and then increased as in the case of hydrogen.

181. Amagat.—Regnault's results have now been superseded by the more extensive work of Amagat, who employed apparatus giving much higher pressures. These were measured by a nitrogen gauge. To construct his scale he built up a steel tube some thousand feet long in the vertical shaft of a mine. This could be filled to any desired height with mercury, so that the resulting pressure at the bottom of the mine could be easily ascertained. His apparatus is shown in the diagram, Fig. 81. In this AB is the vertical pipe; CD the tube in which the nitrogen was contained, S a screw to force the mercury contained in the chamber M up the pipe AB. The nitrogen tube was carefully calibrated, though it does not appear that its expansion under high pressure

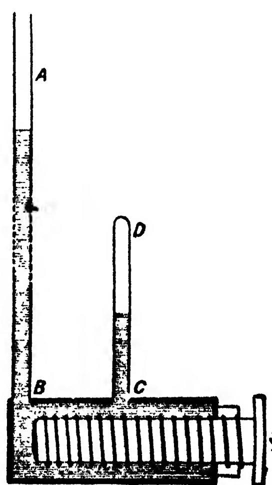


Fig. 81.

was determined. It was also water-jacketed, to ensure a constant temperature.

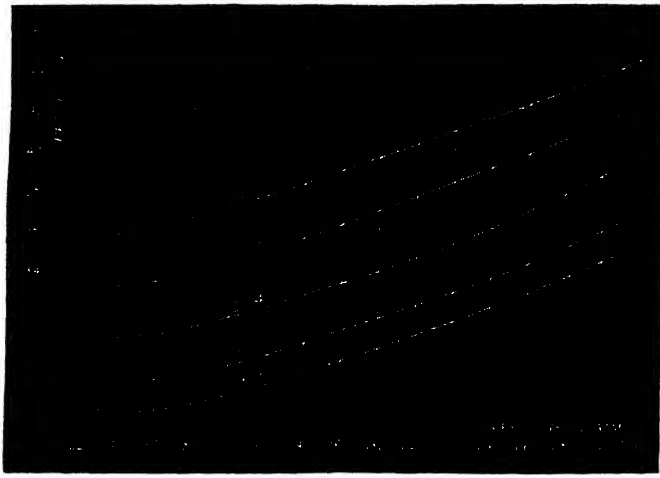


Fig. 82.

Fig. 83.

When the nitrogen manometer had been thus constructed it became possible for him to use it to determine the

pressures of other gases, employing for their compression apparatus similar to that of Regnault. His results were

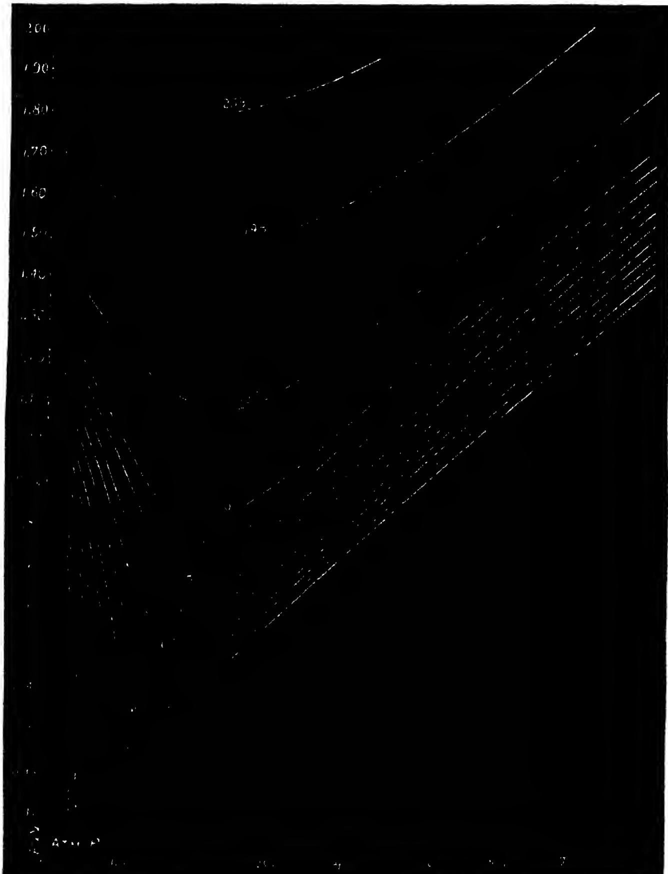


Fig. 84.

exhibited in curves connecting PV (ordinate) with P (abscissa). Some of these are given in Figures 82, 83, 84. To sum up the matter it would appear that—

(1) At low pressures in all gases the product PV decreases as P increases.

(2) At high pressures PV increases with P.

(3) There is therefore a minimum value of PV.

(4) The value of P corresponding to this depends on the nature of the gas and on its temperature, the drop in the curve being most marked near the critical temperature.

182. **Van der Waals' Equation.**—Boyle's Law combined with that of Charles gives rise to the equation $PV = RT$, where R is a constant (*v.* Ex. 1, p. 185) and T is the temperature measured on the absolute scale.

The ideal substance that follows this relation is called a perfect gas: as shown above all actual gases deviate considerably. Many attempts have been made to modify the equation and formulate one nearer the truth. The equation of Van der Waals is the most important. He attempted to take into account both the bulk of the molecules and their mutual attractions.

One effect of attractions between molecules would be to make the internal pressure of a gas greater than the external, for the molecules on the outside would be attracted inwards so that the pressure on the walls would be lessened. Denote this additional pressure by ω. Then if ω is proportional to the product of the masses of gas attracting and attracted, and each of these masses is proportional to the density, it follows that their product is proportional to the square of the density or the inverse square of the volume; we may therefore put $\omega = \frac{a}{v^2}$, where a is some constant independent of temperature.

$$\text{Hence internal pressure} = \text{external pressure} + \frac{a}{v^2}$$

$$= p + \frac{a}{v^2}.$$

The volume in which the molecules are free to move is less than the volume occupied by the gas by some fixed amount equal or proportional to the bulk of the molecules themselves. Call this b, and we arrive at Van der Waals' equation,

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT.$$

This equation involves three constants, a, b and R, which are to be determined by experiment. This has been done for most gases, and the results obtained show fairly close agreement. The law is therefore nearer truth than Boyle's Law unmodified, but is still not quite correct.

183. Molecular Free Path.—As the molecules of a gas move about they frequently come into collision with one another. The average length of path between consecutive encounters and the probable number of encounters per second which a single molecule undergoes may be calculated. We shall not attempt to solve these problems completely, but, to avoid mathematical difficulties, shall take a simple case, the solution of which will indicate the methods by which the general problems have been worked out by Clausius, Maxwell, and others.

Using the notation of Art. 123 let us divide the unit cube, in which there are on an average n molecules, into n equal cubes, each of edge λ: the volume of each of these is λ^3, so that $n\lambda^3 = 1$.

At any instant there will be on the average one molecule in each of these little cubes. λ is termed the mean distance between the molecules.

Each molecule occupies a certain space within which no other molecule can trespass: this space might be the actual volume of the material of the molecule, but it is only necessary to consider that they have a *sphere of action* and that no one of these spheres can ever contain the centre of another. Let S be the cross-sectional area of one of these spheres.

Consider now a single molecule, A, in motion amid others at rest. When A moves forward a distance x, it traces out a volume Sx. If there happens to be a molecule with its centre inside this volume a collision will take place. Now on an average there is one molecule per volume of λ^3. We shall assume that the chance of collision taking place is the ratio of these volumes; this assumption is legitimate if x is a very small quantity. Hence chance of collision in a distance x is $\dfrac{Sx}{\lambda^3} = p$ (say).

Now let us think of a molecules like A, which are very widely separated from one another. These alone are to be considered in motion. If a is very large, then pa will collide in a distance x, $a - pa$ will pass on. Of the $a(1-p)$ molecules which pass on, $pa(1-p)$ will collide in travelling the next distance x, and the remainder $pa(1-p)^2$ will travel on.

Hence pa travel a distance between x and o,

$pa(1-p)$ „ „ „ $2x$ and x,

$pa(1-p^2)$ „ „ „ $3x$ and $2x$,

and so on.

The total length of paths travelled by all the a molecules will therefore be nearly

$$
\begin{aligned}
& pax + pa(1-p)2x + pa(1-p)^2 3x + \ldots \\
&= pax\{1 + 2\,\overline{1-p} + 3\,\overline{1-p}^2 + \ldots\} \\
&= pax(1 - \overline{1-p})^{-2} \\
&= pax\,p^{-2}. \\
&= \frac{ax}{p}.
\end{aligned}
$$

This being the total length passed over by the a molecules before collision, the average length of path before collision is obtained by dividing by a.

Hence average free path $= \dfrac{x}{p} = \dfrac{\lambda^3}{S} = L$ (say).

To solve the actual physical problem we should have to consider that all the molecules are in motion, and that their velocities are seldom equal to the average velocity.

An amended formula is $L = \frac{3}{4}\dfrac{\lambda^3}{S}$

(*v.* Meyer's Kinetic Theory of Gases).

184. Frequency of Collisions.—If the average length of free path of a molecule is L and its velocity V, it will travel on an average for a time $\dfrac{L}{V}$ before collision, so that the number of collisions per second is $\dfrac{V}{L}$.

In Article 192 an attempt is made to evaluate numerically L, λ, S.

185. Viscosity.—If the faces of two blocks of wood are wetted with hot glue and placed in contact, then at first they may be easily moved about over one another. As the glue cools and dries the effort necessary to produce this relative motion grows larger and larger until the two blocks stick tightly together. Let us consider the forces which are acting in the intermediate stages.

Call the blocks A and B; consider B fixed, while A moves with a velocity V. The glue in immediate contact with the faces A B remains there all the time—there is no slipping between the wood and the glue. The slipping takes place in the liquid. Imagine the liquid divided up into a very large number of excessively thin layers parallel to the faces of A and B. Then the bottom layer is at rest in

contact with B while the topmost travels along with A at a velocity V. The velocity of any intermediate layer will lie between these values. The work done by the forces which must be applied to A is expended mainly in producing relative motion between successive layers of glue, so that as these slip over one another frictional forces come into play in the interior of the liquid.

Internal friction or viscosity exists whenever two portions of any fluid, liquid or gas, move with different velocities. It is this viscosity which renders the fall of a mist drop through the air so slow, which causes the waves on the sea to subside when the wind falls, and renders possible the formation of eddies and whirlpools.

186. We shall make the following **assumptions**:—

(1) Viscosity depends only on relative motion.

(2) When a fluid moves over the surface of a solid there is no slip between the solid and the fluid. This assumption is based on experiment and is justifiable except in extraordinary cases, *e.g.* when a body moves in a gas at very low pressure.

(3) That the relative velocity of two parallel layers of a fluid is proportional to the distance between them, provided this distance is small. This assumption is due to Newton. From it, it follows that the velocity of a layer is a linear function of its distance from a fixed parallel plane—in other words, that if v is the velocity of a layer at a distance x from a fixed plane, then $v \propto x$, so that $v = \lambda x$ when λ is a constant quantity independent of x or v.

187. We are now in a position to define the **Coefficient of Viscosity**.

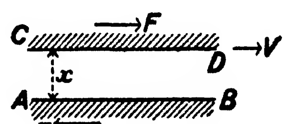


Fig. 85.

Let a layer CD of fluid move with velocity V relative to a parallel layer AB which is at a distance x from it.

Let the force on an area a required to produce this motion be F. On CD this force must act in the direction of V; on AB in the opposite direction.

Then
$$F = \eta \frac{aV}{x},$$

or
$$\eta = \frac{xF}{Va}.$$

η is termed the Coefficient of Viscosity. Its dimensions are

1 in Mass, -1 in Time, -1 in Length.

188. Definition of Viscosity.—The viscosity of a fluid is therefore measured by the tangential force per unit area required to maintain a relative velocity of unity between two parallel planes in the fluid at unit distance apart. This definition applies to liquids and gases. The treatment of viscosity in liquids is deferred to Chapter XIII.

189. The Viscosity of Gases is easily accounted for by the Kinetic Theory as being due to an interchange of molecules between different layers of the gas. For consider a plane CD at a distance x above the fixed plane AB.

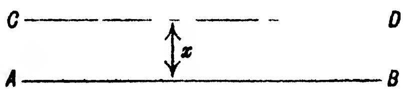


Fig. 86.

Then the velocity v of the layer CD may be taken as λx (*v.* Art. 186). Now across unit area of the face CD a certain number of molecules are being shot each second, some passing upwards and some downwards. The number that pass in either direction is (*cf.* Art. 173) $\frac{nV}{6}$. The number is obtained by assuming that the molecular motion produces the same effect as would be produced by dividing the particles up into three groups—one moving perpendicular to the plane CD, and the other two perpendicular to one another and in the plane of CD. These $\frac{nV}{6}$ molecules come from different places between AB and CD, but on an average the distance that they travel is L, the mean free path. Their average translation velocity is therefore that

of a layer L below CD, and so is equal to $\lambda(x-L)$. They pass then carrying with them a momentum

$$\lambda(x-L)\frac{nV}{6}m$$

in the direction CD.

Coming from the other side the same number of molecules pass, but these pass over with the velocity of a layer $x+L$ from AB, so that the momentum passing downwards is $\lambda(x+L)\frac{nV}{6}m$. The difference between these two measures the total momentum lost by the layers above CD to those below. This difference $=\frac{\lambda LmnV}{3}$.

This loss of momentum is the cause of the drag on the layer CD. In accordance with Newton, Law II., the force is measured by the rate of change of momentum.

Hence F, the force per unit area $=\frac{\lambda}{3}LmnV$.

But
$$F=\eta\frac{v}{x}$$

$$\therefore\quad \eta=\frac{x}{v}\cdot\frac{\lambda}{3}LmnV$$

$$=\tfrac{1}{3}LmnV,$$

$$\eta=\frac{pL}{V} \qquad \text{(Art. 173)}$$

or
$$L=\frac{V\eta}{p}$$

Combining the equation

$$\eta=\tfrac{1}{3}LmnV$$

with those of Art. 183, viz.

$$L=\frac{\lambda^3}{S},\qquad n\lambda^3=1$$

we find
$$\eta=\tfrac{1}{3}\frac{mV}{S}.$$

190. Pressure and Viscosity.—The right hand side of this equation involves no term which is dependent on the pressure. This indicates—if the theory is right—that the viscosity of a gas is independent of its pressure. The result is contrary to general expectation; for it is popularly supposed that the viscosity or internal frictional resistance of a gas decreases as the pressure decreases. Maxwell, therefore, who first obtained the formula, carried out experiments to test its truth. Had it proved experimentally unsound the Kinetic Theory must have been abandoned, or at least considerably modified.

Two important methods have been used to determine the viscosity: the disc method employed by Maxwell and Meyer, and the transpiration method of Poiseuille. In the disc method five plane discs are fixed on the same vertical axle and so suspended that they can make torsional vibrations. Between and very close to these discs are placed other discs, fixed to the vessel containing the gas the viscosity of which is required. The function of the fixed partitions is to prevent the gas as a whole from rotating with the vibrating system.

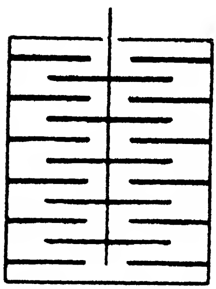


Fig. 87.

Now if a system which without friction would oscillate with simple harmonic motion is subject to a retarding force always proportional to its velocity, the logarithm of the amplitude of vibration gradually decreases at a rate which can be calculated when the retarding force is known. In the disc experiment the logarithmic decrement in the oscillation can be observed and the retarding force calculated.

The results of experiments by Maxwell, Meyer, and deductions from the work of Graham show that the results predicted by theory are not contradicted by experience: viscosity is independent of the density of a gas provided that the pressure is not extremely low.

In a confined space at a high vacuum, *e.g.* in the case of Crookes' radiometer, the viscosity effects are much

reduced. The method of Poiseuille will be considered in Chapter XIII.

191. Temperature Effects.—The value of η given by the equations of Art. 189 indicates that the viscosity of a gas is dependent on V, and therefore increases with the temperature. This result has been verified experimentally.

192. Numerical Values of the Length of Free Path: Frequency of Collision.

The value of η for air is about ·00018.
Substitute this value in the above equation $\eta = \frac{1}{3}LmnV$.
The value of V for air is about 48000.

$$mn = \rho = \cdot 0013.$$

Hence
$$\cdot 00018 = \tfrac{1}{3} \times L \times \cdot 0013 \times 48000.$$
$$\therefore L = 8 \times 10^{-6} \text{ cm.}$$

The frequency of collision
$$= \frac{V}{L} = 5 \times 10^{9}.$$

Both these numbers require certain corrections. When these are made we find that a particle of air moves with a velocity of some 480 metres per second, that it collides with other particles about five thousand million times per second, and that the distance it moves between consecutive collisions is on an average about eight ten-thousandths of a millimetre.

193. Effusion.—If a gas is contained in a thin-walled vessel in which a small hole is made the gas will escape. The process is called effusion. Many experiments on effusion were carried out by Graham. These are described in the *Phil. Trans.*, 1846. He allowed different gases to escape through short glass tubes and small holes in thin brass plates, and noted the times of escape of equal volumes. He came to the conclusion that all gases escaped in such a way that the times for equal volumes of different gases were proportional to the square roots of their densities. In the case of hydrogen there was some little discrepancy, due probably to the wall not being sufficiently thin, so that the gas was stopped by viscosity effects.

Bunsen ("*Méthodes gasométriques*") carried out further experiments. His method was practically to use a barometer tube on the top of which was sealed a metal

diaphragm pierced with a small hole. When this was opened gas entered the tube and the rate at which the mercury fell was noticed. By this method he calculated the densities of different gases. His results agreed with Graham's Law.

The explanation on the kinetic theory is as follows. Suppose the area of the hole is a. Then, as might be expected, the number of molecules that pass into the mouth of the opening is (approximately) proportional to a, the velocity of the molecules (V), and the number (n) per unit volume. Not all of these will pass through, for they will impinge on molecules of gas outside and bounce back again into the vessel. If, however, the external space is vacuous (or even if the external pressure is much less than the interior), there will be nothing from which the molecules can rebound. In this case then we may assume that the number of molecules passing through the hole per second

$$= kVna, \text{ where } k \text{ is some constant.}$$

Hence the mass escaping per second—or rate of efflux—

$$= kVnam, \text{ where } m \text{ is the mass of a molecule,}$$

$$= ka\rho V$$

$$= ka\rho\sqrt{\frac{3p}{\rho}}, \text{ since } V = \sqrt{\frac{3p}{\rho}},$$

$$= ka\sqrt{3p\rho}.$$

If we wish to compare the rates of efflux of two different gases G, G' at the same pressures, we have

$$\frac{\text{Mass per second of } G}{\text{Mass per second of } G'} = \frac{ka\sqrt{3p\rho}}{ka\sqrt{3p\rho'}} = \sqrt{\frac{\rho}{\rho'}},$$

$$\therefore \frac{\text{Volume per second of } G}{\text{Volume per second of } G'} = \sqrt{\frac{\rho'}{\rho}}.$$

In other words, the rates of effusion under similar conditions of different gases are inversely proportional to the square roots of their densities.

194. Transpiration.—If the hole through which the gas flows is fine but not extremely short, the passage of the gas is called transpiration. Some experiments on transpiration are easily performed. Take a dry porous pot such as those employed in Daniell cells, fit it with a cork through which passes a long glass tube. Hold this upright, the cell uppermost, and let the end of the tube dip into a beaker of coloured water. No apparent change will take place. Now surround the jar with an atmosphere of hydrogen or any gas lighter than air. This may conveniently be done by collecting the hydrogen in a large receiver and lowering the open mouth over the pot. The hydrogen at once begins to enter the jar through the pores of the cell and the air inside to leave it. But the molecules of hydrogen move faster than those of air, consequently the hydrogen is shot in faster than the air escapes. The pressure inside the jar rises, gas is expelled and bubbles up through the water in the beaker. Now remove the jar of hydrogen. The gas in the pot is largely composed of hydrogen, which begins to escape faster than air can enter from outside to take its place, so that the pressure falls and the coloured water runs up the glass tube. As all the hydrogen escapes the liquid regains its original level.

Similar experiments can be made with carbon dioxide; the effects, however, will be reversed.

The holes in the walls of the pot approximate to long fine tubes. The rate of flow of fluids through such is dependent on viscosity. Poiseuille has shown how viscosity of a fluid may be measured by its rate of transpiration. His analysis will be found in Chapter XIII. The laws which gases obey appear from Graham's experiments to be much the same as the laws of Effusion. Thus through biscuit-ware and compressed graphite gases transpire, *ceteris paribus*, at rates inversely proportional to the square roots of their densities. There is, however, this important difference, that whereas in Effusion the percentage composition of a mixture of gases is unaltered, in Transpiration each of the components finds its way through independently of the others, at its own pace. As an

example suppose equal volumes of hydrogen and oxygen are contained in a thin brass vessel in the top of which there is a small hole. Then after any time the volume of hydrogen that has escaped will be equal to that of the oxygen. If, however, the gases are in an unglazed pot and surrounded by the atmosphere, then more hydrogen will pass out in a second than oxygen, so that the composition is changed, the gas that has escaped being richer in hydrogen, that which is left richer in oxygen, while some air will enter. This partial separation of mixed gases is sometimes useful. To separate them they may be passed through the stem of a long churchwarden pipe. If the gas at first contains equal parts of hydrogen and oxygen, then the composition of the gas that transpires will be four of hydrogen to one of oxygen, the ratio $4:1$ being that of $1:\frac{1}{\sqrt{16}}$, *i.e.* that of the square roots of the densities. The gas might then be collected and again passed through a pipe to obtain a mixture still richer in hydrogen. The method might be profitably employed to extract oxygen from the air, were it not that the density of oxygen in practically the same as that of nitrogen.

[NOTE.—Some writers restrict the use of the word transpiration to the case of a fluid passing through a tube which is not extremely fine. Such a tube does not separate the components of a mixture of gases. These writers speak of the passage of a gas through a porous pot as diffusion.]

Graham defines effusion as the passage of a gas through an aperture in a thin plate: transpiration as the passage through a tube. Some of his experimental results are given in the following form:—

Different gases pass through minute apertures into vacuum in times which are as the square roots of their respective specific gravities—or with velocities which are inversely as the square roots of their specific gravities; that is according to the same law as gases diffuse into each other.

For equal volumes of air of different densities the times of transpiration are inversely as the densities. This

distinguishes effusion from transpiration, for air of all densities passes into a vacuum by effusion with equal velocity.

The time for the effusion of air of different temperatures is proportional to the square root of its density at each temperature.

195. Diffusion.—If a little bromine is dropped into a tall glass cylinder it will evaporate and form a deep brown vapour which occupies the bottom of the cylinder. If it be left for a little time undisturbed, the vapour will gradually spread out and rise higher and higher in the cylinder, while at the same time air will descend to the bottom, mixing with and diluting the bromine vapour.

These effects can be detected by the colour of the contents of the cylinder. If the cylinder is closed at the top and left a long time, the two gases interpenetrate so intimately that the mixture becomes homogeneous: no difference can be detected between the top portion and the bottom. The process by which this mixing is accomplished is termed diffusion. It will be noticed that in the instance we have cited the heavier gas (bromine vapour) rises up against gravity, while the lighter air descends.

If the cylinder originally contained hydrogen instead of air, similar effects would be observed, but the diffusion would be seen to be more rapid. The velocity of diffusion is therefore dependent on the nature of the gases. The kinetic theory explains readily the interdiffusion of two gases. The extreme slowness is due to the shortness of the free path of the molecules. If the pressure be diminished the length of free path is increased, and diffusion takes place more rapidly. Further treatment of diffusion is deferred to Chapter XIII., where the phenomenon of liquid diffusion is discussed.

196. Solution of Gases in Liquids.—It is generally known that gases are soluble in liquids. The air is dissolved by water, and in this form is breathed by fish.

When water after being shaken up with air is heated in a glass vessel small bubbles of gas make their appearance

on the sides of the vessel. If this be collected in a test tube in sufficient quantity, it will be found that it is capable of rekindling a glowing match. This shows that the gas is richer in oxygen than air.

Two conclusions can be drawn from this experiment: (1) oxygen and nitrogen have different solubilities in water, (2) air is less soluble in hot water than in cold. From (2) it again follows, in accordance with thermodynamic principles, that when air is dissolved in water the solution is accompanied by a rise in temperature.

Gaseous solutions are divisible into two classes · the first in which the gas may be entirely removed by repeated distillation (*e.g* solution of air in water); the second in which some kind of chemical action seems to take place which prevents such separation. This class of solution is well illustrated by solutions of hydrochloric acid.

If a very weak solution of hydrochloric acid is boiled, water at first passes over in relatively larger quantities than the gas, so that the concentration of the remainder increases until it has reached a certain fixed value (20 per cent.), when the two components begin to pass over together without alteration. As concentration increases the boiling point gradually rises from a little above 100° C. to a final value of 110° C.

A strong solution of acid behaves in a similar way. The gas passes over at first in large quantities, the solution becomes weaker, the boiling point rises till the solution attains the same final state as that of the weak solution already considered. The final composition of the solution and its boiling point depend on the height of the barometer.

197. Henry's Law.—The solubility of gases in liquids has been carefully studied by Henry (1803). His experiments, subsequently confirmed by Bunsen, led to the conclusion that the volume of a gas which can be absorbed by any liquid is independent of the pressure: or, in other words, the mass of a gas which will dissolve in a definite quantity of liquid is directly proportional to the pressure.

The law does not hold even approximately with active gases such as hydrochloric acid, or with others at high

pressure. The solubility of a gas varies greatly with its nature. Thus the solubility of carbonic acid at 0° C. is nearly 100 times that of hydrogen. By the solubility of a gas is meant the ratio of the volume of gas dissolved to the volume of the solvent.

Unless chemical actions take place the addition of a soluble solid to a liquid decreases its power of solution. Thus if sugar is added to a glass of soda water carbonic acid is thrown out of solution, a copious stream of gas being evolved.

In the case of a mixture of gases (*e.g.* ordinary air) each constituent dissolves independently of the others, the mass dissolved being proportional to the pressure itself exerts. Thus in the case of air the pressure of the nitrogen is four times that of the oxygen, and the solubility is about half. Hence for every two litres of nitrogen absorbed by water from the air one litre of oxygen is absorbed.

198. **Passage of Gases through Solids.**—Many solids have the power of absorbing gases on their surfaces. In making a barometer it is necessary to carefully free the glass tube from air which condenses on its sides. Charcoal readily absorbs many gases, which can be expelled by heating. The effects of palladium and spongy platinum on hydrogen are well known. In some cases the absorption appears to be of a chemical nature: it has been supposed that hydrogen forms an alloy or loose compound with palladium.

Sheet rubber has this power of absorption. If it is used to separate air from hydrogen, the hydrogen on one side of the sheet is absorbed, passes through, and is given up on the other. It may be that the rubber forms a kind of chemical compound with the hydrogen, stable only where the pressure of the gas is comparatively great. In this case the compound would be formed on the one side and decomposed on the other. This passage of gas accounts for the fact that rubber balloons filled with hydrogen quickly collapse. A soap bubble filled with a gas fairly soluble in water (*e.g.* carbonic acid) behaves in the same way.

Iron at high temperature is fairly permeable to carbon monoxide. Hence coke stoves at a red heat discharge this gas into the air of a room in which they burn. In this case a definite compound is probably formed.

Fig. 88.

199. Measuring Instruments.—The Siphon Barometer is essentially a U-tube, one limb of which (*EC*) is open to the atmosphere, and the other (*CA*), which is much longer, is sealed at the top.

The part *BCD* contains mercury. The space *AB* is a vacuum except for the unavoidable mercury vapour.

The pressure at *D* is atmospheric, that at *F* on the same horizontal line must be the same.

But the pressure at *F* is that due to the mercury column, *BF*.

Hence the pressure of the atmosphere is measured by the difference in level between the two free surfaces at *B* and *D*.

Scales are either engraved on the glass or are placed behind the parts of the tube near *B* and *D*. These scales, reading to fractions of inch or centimetre, are generally so placed that their zeros are on the same level. That for the part near *D* reads downwards, while that for *B* reads upwards. The difference in level is easily found by adding together the two numbers opposite to *D*, *B*. Thus in Fig. 88 the height of the barometer is 26 + 4, *i.e.* 30 inches of mercury.

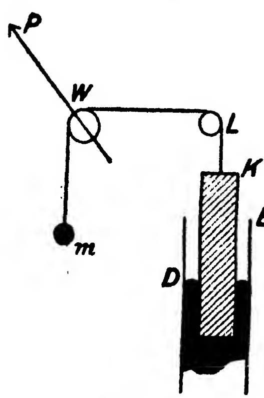


Fig. 89.

A modification of this instrument is in frequent use (Fig. 89). The limb *DE* is fairly wide, and a heavy iron cylinder (*K*) floats in the mercury. This is

attached to a string which passes over a pulley, *L*, is wrapped round and fastened to a wheel, *W*, and supports a little weight, *m*. On the axle of the wheel *W* is fixed a long pointer, *P*, which moves over a scale on the face of the barometer. If the pressure of the atmosphere falls, the mercury at *D* rises and lifts the cylinder *K*. This allows *m* to pull the pointer *P* round to the left. For a rise in pressure *P* would move to the right.

High and low pressures are sometimes accompanied by fine and wet weather respectively. Hence a barometer is popularly used as a weather glass, and the left-hand side of the scale is often marked "Rain," the right-hand side "Set fair."

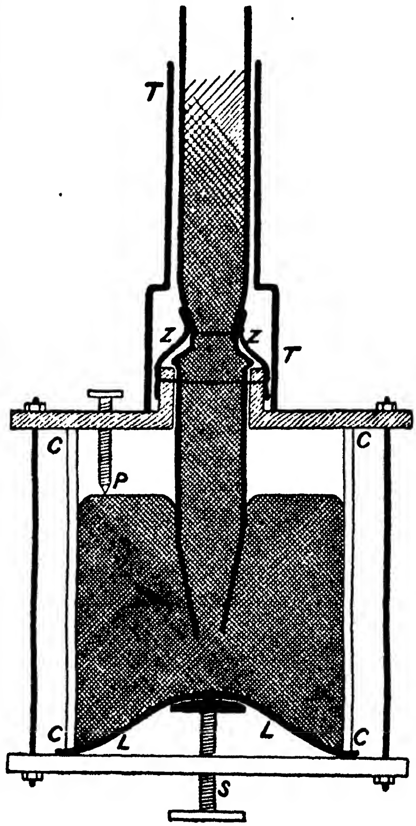


Fig. 90.

200. Fortin's Barometer.—On the siphon barometer two readings are required before the pressure of the atmosphere can be determined. Neither of these is susceptible of great accuracy. Instruments for scientific work are usually made from a straight glass tube sealed at the top; its lower end is immersed in a bowl of mercury. If this bowl or reservoir is fairly large,

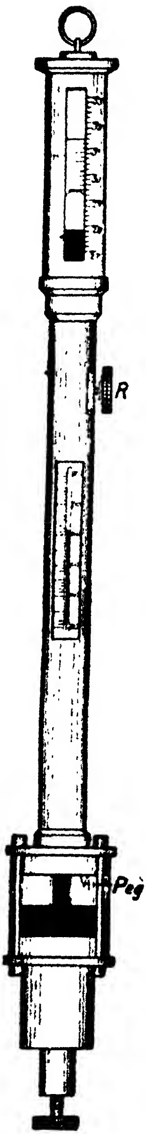


Fig. 91.

alterations in the height of the column will make little difference in the level of the mercury in the bowl. One reading, therefore, is alone required.

In Fortin's barometer (Figs. 90, 91) the mercury reservoir is a short glass cylinder, *CCCC*. This is closed at the bottom with a piece of chamois leather, *LL*, permeable to air but not to mercury.

The leather is secured to the glass, but is supported by a screw, *S*. This screw should be so adjusted that the surface of the mercury is just pushed up to meet a spike, *P*, which is fixed to the top or lid of the reservoir. The level of the mercury in the reservoir is thus always brought to a definite point before the reading is taken. The lid does not fit tightly, but is open to the air. Should the instrument be deranged the mercury is kept from escaping by a piece of leather, *ZZ*. The tube dips deep into the reservoir and the part above is cased in by a brass tube, *TT*. At the top are two glass windows opposite to one another, through which the level of the mercury can be examined.

One of the edges of the front window is graduated (inches or centimetres), and in the window slides a brass vernier the height of which can be regulated by the screw *R*. In the window at the back there is another slide which moves up and down with the one in front, their lower edges being always in the same horizontal plane. To take a reading the observer always keeps his eye in this plane, *i.e.* he puts his eye in such a position that he can just, and only just, see the two edges. He then turns the screw near the top and lowers the vernier till the plane of the edges is tangential

to the protuberant surface of the mercury. A piece of white glass is usually placed behind the barometer to enable this position to be clearly seen. The reading of the barometric height is given by the scale and vernier. This instrument is usually supported by gimbals, so that it may swing with its axis vertical.

In a properly made instrument the mercury cannot escape however it is tilted, for the main tube dips deep into the reservoir, while the leather at the top and bottom prevents any overflow.

The pressure at a depth h in a liquid of density ρ is given by the relation $p = g\rho h$.

If h is measured in centimetres, ρ in grammes per c.c., then p is expressed in dynes per square centimetre. Now as g, ρ are almost constant, it follows that $p \propto h$. Hence it is usual to speak of the pressure as being equivalent to some number of inches or centimetres of mercury. When great accuracy is required this number must be reduced to the number that the barometer would indicate were the density of the mercury equal to that of pure mercury at standard temperature, and the acceleration due to gravity equal to that at some fixed place.

Mercury is easily obtained pure, so that its density is dependent solely on its temperature. The temperature affects the reading in another way, for the brass scale expands when heated. Another source of error is due to capillarity. This is explained in Chapter XV., where it is pointed out that the mercury surface is depressed by a calculable amount.

201. Correction of Barometric Reading.—To show how corrections are applied we will solve the following problem :—

A barometer at X reads 750·0 mm. when the temperature is 25° C. The barometer is correct at 0° C. The scale is of brass. The diameter of the glass tube is ·6 cm. Find what the pressure would be in inches of mercury at Kew at a temperature 15° C.

Coefficient of linear expansion of brass = a. Coefficient of cubical expansion of mercury = β. Density of mercury at 0° C. = ρ.

(1) Correction for scale—

A length l at 0° C. becomes $l(1 + at)$ at t° C.,

$\therefore$ a length 75 cm. at 0° C. becomes $75(1 + 25a)$ cm. at t° C.,

$\therefore$ height of mercury column at 0° C. = $75(1 + 25a)$.

(2) Correction for density of mercury—

If a mass M of mercury has a volume v_0 and density ρ_0 at 0° C. and v_t, ρ_t at t° C., then

$$v_0\rho_0 = M = v_t\rho_t$$

$$\therefore\ v_0\rho_0 = v_0(1+\beta t)\rho_t,$$

$$\therefore\ \rho_t = \frac{\rho_0}{1+\beta t} = \rho_0(1-\beta t).$$

Hence density of mercury at 25° C. is $\rho(1-25\beta)$.

(3) Capillarity correction—

The reason for the depression of the mercury is explained in Chapter XV. The amount of the depression is dependent on the width of the tube and the nature of the glass.

For a tube ·6 cm. diameter the depression is about ·091 cm.

(4) Correction for gravity—

At Kew acceleration due to gravity $= g$.

At X acceleration due to gravity $= g'$.

At X pressure is $= g'.\rho(1-25\beta)\,(75{\cdot}00 + {\cdot}09)\,(1+25\alpha)$.

If x cms. is true barometric height on the Kew barometer, then

$$gx\rho(1-15\beta) = g'\rho\,(1-25\beta)\,(75{\cdot}00+{\cdot}09)\,(1+25\alpha),$$

$$\therefore\ x = \frac{g'}{g}\left(\frac{1-25\beta}{1-15\beta}\right)(75{\cdot}09)\,(1+25\alpha)$$

$$= \frac{g'}{g}(1-10\beta+25\alpha)\,75{\cdot}09.$$

202. Pressure Gauges.—(1) To measure pressures nearly atmospheric it is generally sufficient to use a glass tube as shown in Fig. 92. The bend in this U tube may be filled with any convenient liquid. For an ordinary gas supply water can be used. The height A above $B(=h)$ measures the pressure above atmospheric. Thus, if $h = 5$ cm. and the barometer stands at 760, the pressure of the supply is $\left(760 + \frac{50}{13{\cdot}6}\right)$ mm. of mercury.

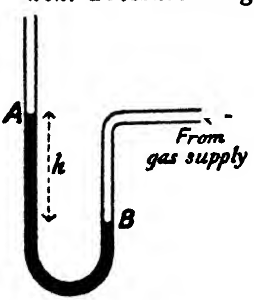


Fig. 92.

If the tube is filled with mercury the gauge may be used to measure between, say, ½ and 1½ atmospheres.

203. (2) For high pressures a convenient instrument is an air manometer (Fig. 93). This is a U tube with the bend filled with some liquid. The limb A contains air, and is graduated so that the pressure, which is inversely proportional to the volume of the air in A, can be calculated or read off from a scale.

Fig. 93.

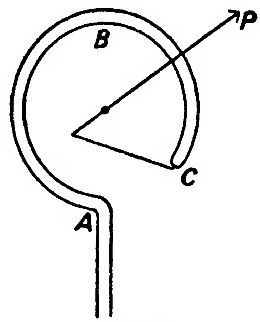


Fig. 94.

204. (3) **The Bourdon Gauge** can be used for pressures above and below atmospheric. It consists of a tube ABC, elliptical in section, closed at C, and opening at A to the supply (Fig. 94).

If the pressure of the supply is increased the tube tends to become more circular in section. To do this the end C must be forced away from A, so that the pointer P to which it is connected moves over the scale. The instrument is graduated by comparing its indications with a standard gauge. The aneroid barometer is a modification of this instrument.

205. (4) **McLeod Vacuum Gauge.**—For measuring very low pressures, *e.g.* that of the residual gas in an electric incandescent lamp, a McLeod gauge, shown in the diagram, Fig. 95, may be used. AB is a narrow tube joined to the vessel BC. Into this vessel are led two pipes, one DE from the supply, the other FG from a well of mercury G. To use the instrument, G is lowered till BC and DE are both

empty. The tube AB then communicates directly with the supply, and so contains the same gas as the supply and at the same pressure. The well G is then raised, mercury enters, closes the bottom of the tube DE, and drives the gas from BC into the narrow tube AB. The pressure of the gas in this tube is read off as the barometric height, less the depth of G below B. From a knowledge of this pressure and the volumes of AB and BC, the pressure of the gas which originally occupied the whole of AB and BC can be calculated, if we assume Boyle's Law to hold. Gases, however, are apt to condense on the glass. In such cases the law cannot be applied, and the instrument fails to give concordant results.

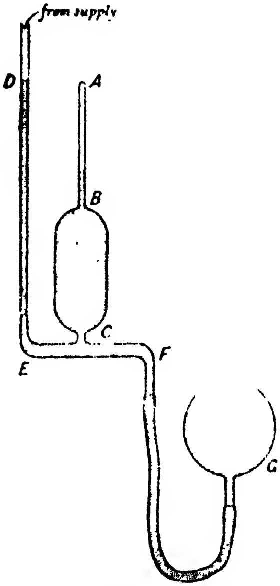


Fig. 95.

206. Relations between Pressure and Height.—Take two points P and Q near one another in the atmosphere, Q being vertically above P. Call the heights of P and Q above the ground x and $x + dx$, and let the pressures at these points be p, $p + dp$, and the density of the air ρ and $\rho + d\rho$.

Let a small cylinder be described about the vertical line PQ as an axis of unit sectional area.

Resolve vertically for the equilibrium of this cylinder

$$(p + dp)a + g\rho dx a = pa,$$

$$\therefore\ dp + g\rho dx = o.$$

If the air is at constant temperature

$$\rho \propto p$$

$$= \lambda p \text{ say,}$$

$$\therefore\ \frac{dp}{p} + g\,.\,\lambda dx = o.$$

Integrate; $\log_e p + g\lambda x = \text{constant}.$

Hence if p_1, p_2 be the pressures at heights h_1, h_2, then

$$\log p_1 + g\lambda h_1 = \log p_2 + g\lambda h_2,$$

i.e.
$$\log_e \frac{p_1}{p_2} = g\lambda(h_2 - h_1).$$

In this equation $\lambda = \frac{\rho}{p}$

$$= \frac{\cdot 0013}{76 \times 13\cdot 6 \times 980}.$$

Also $\log_e \frac{p_1}{p_2} = \mu \log_{10} \frac{p_1}{p_2}$, where $\mu = 2\cdot 302$.

Hence
$$\log_{10} \frac{p_1}{p_2} = \frac{g}{\mu} \lambda(h_2 - h_1).$$

The above relation has been obtained on the supposition that the temperature throughout is constant. In actual practice this is not usually the case. The result is, however, accurate enough for determining small heights.

A nearer approximation is obtained by the formula

$$(h_2 - h_1) = 1840000 \left(1 + \frac{t}{273}\right) \log \frac{p_1}{p_2},$$

where t is the mean of the temperatures (C.) of the two places, and h_2, h_1 are supposed expressed in centimetres.

Examples XI.

1. What is meant by the equation $PV = RT$? Find a numerical value for R in the case of any gas.

2. A closed porous pot filled with air is provided with a manometer. Describe what are the indications of the manometer if the pot is suddenly surrounded by and kept in (*a*) coal gas, (*b*) carbonic acid.

Give some explanation of the Kinetic Theory.

3. Show that if a series of heights be taken in *A.P*, the barometric pressures are in *G.P*.

4. Gas burners are sometimes made with cotton-wool stuffed into the passage, thus reducing the flow of gas past the burner; in others the flow is determined by the passage of the gas through a small circular hole. How will the uniformity of consumption of gas in the two cases compare with the variation of gas pressure?

5. The tube of a barometer has an internal area of cross section of ·3 sq. cms. and a height above the surface of the mercury in the reservoir of 8·6 cms. A bubble of air, the volume of which at the bottom of the column is 5 cubic millimetres, is allowed to escape up the tube. Calculate the error it will cause in the reading of the barometer at the normal atmospheric pressure.

6. Explain the construction and theory of a mercury barometer. Show how a knowledge of the distribution of barometric pressure over a given area gives a means of approximately estimating the direction and strength of the wind.

7. Will the upward force on a balloon change as the balloon rises, (*a*) if it be initially only partially inflated, (*b*) if it be fully inflated, but with an aperture below through which the gas can escape?

8. It is said that a heavy gas like carbonic acid can be poured like water from one vessel to another, the air being displaced. It is also said that, left to itself, a mixture of gases such as air and carbonic acid will by diffusion become of uniform composition throughout its volume. Show how far these two statements can be reconciled.

9. Describe the barometer and explain what is meant by its Temperature Correction. A syphon barometer is so constructed that the long closed tube has an internal sectional area equal to $\frac{1}{4}$ of an inch, while the short open tube has an internal sectional area equal to $\frac{1}{2}$ an inch. Find what will take place in the long tube of this barometer when the true pressure of the air falls one inch.

10. Describe a form of barometer suitable for accurately measuring the atmospheric pressure. Find the height of the barometer reduced to zero when the reading at $t°$ C. is h, the scale being of brass. (Coefficient of absolute expansion of mercury is 0·00018; coefficient of linear expansion of brass is 0·000018.)

11. Describe an experiment to verify Boyle's Law for pressures less than the atmospheric pressure.

12. Some air is in the space above the mercury in a barometer of which the tube is uniform. When the mercury stands at 29 inches in the tube the space above the mercury is 4 inches long. The tube is then pushed down into the cistern so that the space above the mercury is only 2 inches long, and now the mercury stands at 28 inches. At what height would it stand in a perfect barometer?

13. State the laws of diffusion of gases through a plug of porous material. A mixture of hydrogen and oxygen in equal parts by volume is contained (1) in a vessel in which there is a porous plug, (2) in a vessel in which there is a hole, say 1 mm. in diameter; the mixture of hydrogen and oxygen is allowed to escape into a vacuum. How will the proportion of hydrogen and oxygen be affected, if at all, when the escape has been going on for a short time?

14. Determine the height of the barometer when a milligramme of air at 27° C. occupies a volume of 20 cub. cm. in a tube over mercury, the mercury standing 73 cm. higher inside the tube than

outside. (1 gramme of air at 0° C. under pressure of 76 cm. of mercury measures 773·4 cub. cm.)

15. State the law of diffusion of gases through a porous partition. Sketch carefully an apparatus by which it can be shown experimentally that different gases diffuse at different rates, and describe how it is to be used.

16. Prove the following rule for correcting the barometer for temperature:

$$\text{Height at } 0^\circ \text{ C.} = \text{observed height at } t^\circ \text{ C.} \times \frac{5550}{5550 + t}(1 + kt),$$

where k is the coefficient of linear expansion of the scale.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER XII.

HYDROSTATICS.

207. Thrust and Pressure.—A fluid has been defined in Art. 27 as being matter in such a state that it is incapable of resisting permanently a shearing stress. Hence in hydrostatics, *i.e.* the mechanics of fluids at rest, the only forces in existence between two portions of a fluid are forces everywhere normal to their inter-surface. To make matters definite consider two portions A and B of a fluid at rest; let these portions touch one another over a plane area S. Then A will exert some force F on B, B will exert on A an exactly equal opposite force. This force F must, from what we have said above, be normal to the area S. It is called the Thrust on the area S. Hence thrust is force and must have the same dimensions, $M^1L^1T^{-2}$.

Now let us divide the area S into a large number, n, of small areas each equal to a. On each of these areas a a thrust is exerted. These different thrusts may or may not be equal.

Let us first suppose that they are, and that each is equal to T.

Then $$nT = F.$$

Let $$\frac{T}{a} = p.$$

Then $$\frac{F}{S} = \frac{nT}{na} = \frac{T}{a} = p.$$

The quantity p is termed the pressure at any point of the plane S.

If the thrusts on the areas a are not all equal, let us consider a particular area (a) which surrounds any selected point Q. Call the thrust on this area τ. Then $\Sigma\tau = F$.

The value of τ will depend on the size and position of the area a. If, however, n is very large and consequently a very small, the quantity $\frac{\tau}{a}$ will not be dependent on the shape of the area, or its size, for τ and a diminish together. The quantity $\frac{\tau}{a}$ is called the pressure at the point Q. If we denote this by p, then $\tau = ap$. So that $\Sigma a = F/p$. Hence pressure at any point Q in a fluid may be defined as follows:—Surround Q by a small area a; suppose T is the thrust on a. Then the limiting value of the quantity $\frac{T}{a}$ when a is made indefinitely small is termed the pressure at the point Q (cf. definition of surface density in Static Electricity). Since $p =$ force $\div$ area, its dimensions are $M\,L^{-1}, T^{-2}$.

The terms *whole pressure*, *resultant pressure*, are sometimes used. The whole pressure (or thrust) on a surface is the sum total of all the *thrusts* which act on the small elements into which we may suppose the surface to be divided: these thrusts are added together without consideration of the direction in which they act. Thus we may write whole pressure $=\int pds$, where p is the pressure on an element ds of the surface. The term whole pressure, however, is useless.

In the resultant pressure the direction of the pressure is taken into account. It is a single *force*, the resultant of all the thrusts on the several elements. Thus, in the case of a floating body, the resultant pressure is equal to the weight of the body, but is directed in the opposite sense.

208. The pressure at a point is the same in all directions.

Take any point O in the fluid.

Take any three directions OX, OY, OZ at right angles.

Mark off on the lines OX, OY, OZ *short* lengths Ox, Oy, Oz each equal to a. Join zx.

Complete the prism $Oxyzz'x'$.

Call the pressures in the directions OX, OY, OZ, and perpendicular to the face $zz'x'x$, p_x, p_y, p_z and p respectively. The forces acting on the prisms are the thrusts on the five faces and the weight. Now the weight is proportional to the volume, *i.e.* is proportional to a^3. The thrusts are proportional to the areas of the faces, *i.e.* to a^2. If then

a is a very small quantity, a^3 can be neglected compared with a^2, *i.e.* the weight (or other forces dependent on the volume) can be neglected compared with the thrusts.

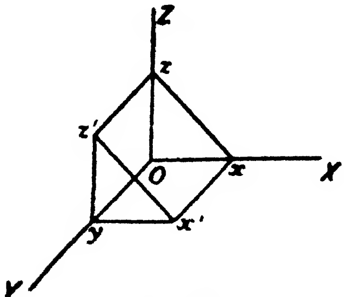


Fig. 96.

The thrusts on the face

$$zxx'z' = p \times area$$
$$= p \times xx' \times xz$$
$$= pa^2\sqrt{2}.$$

It is inclined at 45° to OX. Hence its component along OX

$$= pa^2\sqrt{2}\cos 45° = pa^2.$$

The thrust on the face $zOyz'$ is equal to $p_x \,.\, a^2$, and is parallel to OX. These two thrusts are the only forces which have components along OX, hence equating them

we have $$p_x a^2 = pa^2,$$
$$\therefore p_x = p;$$
similarly $$p_z = p,$$
$$\therefore p_x = p_z.$$

Now OX, OY were any two directions at right angles, hence the pressures at a point in any two directions at right angles are the same.

Now take $O\alpha$, $O\beta$ any two directions. Draw $O\gamma$ perpendicular to both. Then the pressures in the directions $O\alpha$, $O\beta$ are both equal to the pressures along $O\gamma$, and therefore equal to one another. Hence the pressure is the same in all directions.

209. The pressure is the same at all points in a horizontal plane which can be joined by horizontal lines in the fluid.

Suppose A and B are such points.

Join AB. About AB describe a very thin cylinder of sectional area a. Call the pressure at A and B p_a and p_b respectively.

The cylinder is in equilibrium and the forces acting on it are

(1) Its weight.

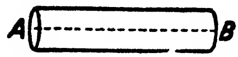


Fig. 97.

(2) The thrusts on the curved surface: these are everywhere perpendicular to the surface, and therefore to the axis AB.

(3) The thrusts $p_a . a$, $p_b . a$ on the ends.

Resolve parallel to AB and we get

$$p_a . a - p_b . a = 0.$$

$$\therefore p_a = p_b,$$

i.e. the pressures at A and B are equal.

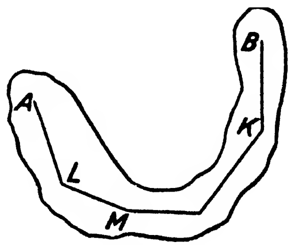


Fig. 98.

If A and B cannot be joined by a straight line lying in the liquid, join them by a zigzag line AL, LM, ... KB, horizontal and lying entirely in the fluid.

Then $$p_a = p_l = p_m = \ldots\ldots = p_b.$$

210. The difference between the pressures at the points in a liquid is proportional to the vertical distance between them, provided that the parts can be joined by a line lying wholly in the liquid.

Fig. 99.

Let A be a point vertically above B.

Surround AB with a cylinder as in Art. 209.

Resolve vertically.

Then $$p_a . a + w - p_b a = 0,$$

where $$w = \text{weight of cylinder}$$
$$= g\rho a h,$$

h being the height of A above B and ρ the density of the fluid.

$$\therefore p_b - p_a = g\rho h,$$

i.e. difference in pressure $\propto h$.

If A is not vertically above B join A and B by a zigzag line.

$$AK, KL \dots NB. \qquad \text{(Fig. 100.)}$$

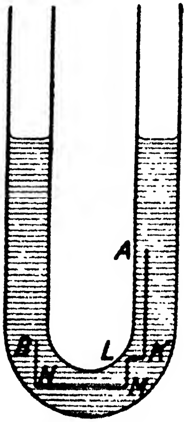


Fig. 100.

Then

$$\begin{aligned} p_a &= p_k - g\rho AK \\ &= p_l - g\rho AK \\ &= p_m - g\rho (AK + LM) \\ &= p_b - g\rho (AK + LM \dots - BN) \\ &= p_b - g\rho h. \end{aligned}$$

This assumes that the points A and B can be joined by a line lying wholly in the fluid. Otherwise the proposition is not necessarily true. Thus the pressure at C (Fig. 101) is not equal to that at D.

211. P = gρh.—If A is a point in the surface of a liquid not subject to external pressure, then $p_a = 0$. Hence the pressure at a depth h in a liquid of uniform density ρ is given by the relation $p = g\rho h$.

On the C.G.S. system p is measured in dynes per square centimetre, g in cm. sec.$^{-2}$, ρ in grm. per c.c., h in cm.

If we measure p in grammes weight per sq. cm., as is more usual, the relation is $p = \rho h$.

Hence the pressure at a depth h in water is given by the relation $p = h$.

We may therefore speak of a pressure of the atmosphere as being equal to that of 76 cm. of mercury, or $76 \times 13{\cdot}6$ cm. of water.

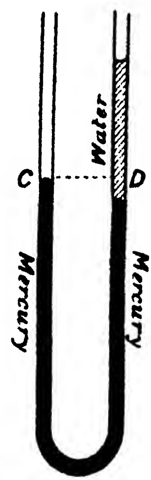


Fig. 101.

The above results are only true for liquids at rest. In the case of liquids in motion the pressure is, *ceteris paribus*, greatest at points where the velocity is least, and least where the velocity is greatest. Thus suppose water flows along a horizontal pipe ABC which has a constriction at B. Then at B the velocity of the water is greater than

at A or C. The pressure at B is therefore less than at A or C. The pressures can be registered by manometers. Such a device is often used to measure the rate of flow of water from a reservoir.

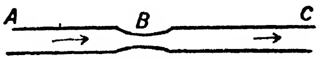


Fig. 102.

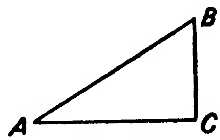


Fig. 103.

212. The surface of a liquid at rest is horizontal; for if not suppose AB is a surface exposed to a uniform pressure P. Then (Fig. 103)

$$P = p_a = p_c = p_b + g\rho\, BC = P + g\rho\, BC,$$

$\therefore BC = 0$, *i.e.* the surface AB must be horizontal.

213. Thrust on a Plane Surface.—If a surface is immersed in a fluid a thrust is exerted on it. If the surface is plane the thrust may be calculated as follows:—Let S be the area of the surface. Divide it up into very small portions. Consider one of these portions, call its area a and its depth h. The thrust exerted on this portion is $ag\rho h$, ρ being the density of the fluid.

The thrust is perpendicular to the area. The thrust on every other element of area is in the same direction. The thrusts may therefore be added together.

Hence total thrust = sum of thrusts on areas like a

$$= \Sigma\, ag\rho h$$
$$= g\rho\, \Sigma ah.$$

Suppose h' is the depth of the centre of gravity of the plane area.

Then $$h' = \frac{\Sigma ah}{\Sigma a}$$

Hence thrust $$= g\rho\, \Sigma ah$$
$$= g\rho h'\, \Sigma a$$
$$= g\rho h' \times \text{total area of surface.}$$

The thrust is therefore equal to the pressure at the centre of gravity of the area multiplied by the area.

214. Upthrust on a Surface.—The vertical component of the thrust on a plane area is equal to the weight of the fluid that could be enclosed between the area and its projection on the free surface. This may be proved by considering the equilibrium of such a volume of fluid. The only forces acting on it are (1) its weight, (2) horizontal thrusts on the vertical sides, (3) the thrust on the bottom. The vertical component of (3) must therefore be equal to the weight. This result may be extended to a surface of any shape if we define the total upthrust on the surface as the sum of the vertical components of the thrusts exerted on elements which make up the surface.

215. Floating Bodies. Archimedes' Principle.—A floating body is acted on by two sets of forces. The one set makes up its weight, the other the thrust of the fluid on it. These two sets of forces must be in equilibrium.

From the last article we can see that the upthrust on the body must be equal to the weight of the fluid displaced by the body; or the argument may be stated thus: suppose the body could be removed without disturbing the fluid in any way, and the hole left filled up with fluid. There would now be equilibrium. The forces acting on the new fluid filling up the hole are (1) the weight, (2) the resultant thrust of the old fluid. These are equal. The thrust of the fluid must be the same whatever it be that fills up the hole. Hence the thrust exerted on the floating body must be equal to the weight of fluid filling up the hole. In other words, if a body floats in a fluid its weight is equal to the weight of fluid it displaces.

The same line of reasoning shows that if a body is totally immersed in fluid, the upthrust of the fluid on the body is equal to the weight of the fluid displaced.

216. Conditions for the Equilibrium of a Floating Body.—The necessary conditions are :

(1) The weight of fluid displaced must be equal to the weight of the floating body.

(2) The centre of gravity of the fluid displaced must be in the same vertical line as the centre of gravity of the

floating body. These two conditions follow directly from the previous articles. Though necessary for equilibrium they are not sufficient for stability. For stable equilibrium we must have

(3) The metacentre must be above the centre of gravity of the body.

217. The Metacentre.—If a body floats in equilibrium in a fluid the centres of gravity of the fluid and body are in the same vertical line. Call these points H, G. Let the line HG be considered fixed in the body. Displace the body slightly by turning it through a small angle, but in such a way that the mass of the fluid displaced remains constant. Then if the vertical through the centre of gravity (H') of the new fluid displaced intersect HG in its new position, the point of intersection is called the metacentre (M) for the displacement that has taken place.

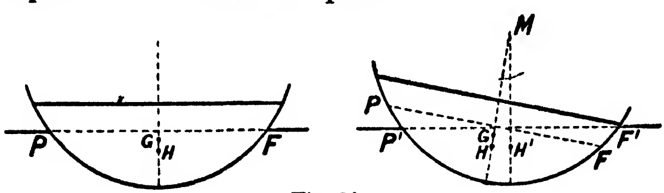


Fig. 104.

The lines actually intersect if the body is symmetrical about the plane of displacement, or a vertical plane perpendicular to it, and in certain other cases. Equilibrium is only stable if M is above G.

The point H is called the *centre of buoyancy*.

The plane PF in which the fluid meets the surface of the floating body is called the plane of flotation.

The two planes of flotation—the old PF and the displaced $P'F'$ meet in a line which we shall call the axis. Then it can be shown that

$$HM = \frac{Ak^2}{\text{volume displaced}}$$

where Ak^2 is the moment of inertia of the plane of flotation about the axis. The fluid is in this case supposed homogeneous.

218. Centre of Pressure.—When a plane area is immersed in fluid the thrust on it is equal to the pressure at the centre of gravity multiplied by the area, but the point of application of the force required to balance the thrust is generally not the centre of gravity of the area. The point is called the centre of pressure. It is defined thus: Divide the area up into a large number of small elements: thrusts are exerted on all these elements—constituting a system of parallel forces. The resultant of these forces is the thrust on the surface and the point in which its line of action meets the area is the centre of pressure.

219. Position of the Centre of Pressure.—Suppose AB is any plane surface of area S immersed vertically in a fluid of density ρ. Divide up into small parts. Call one of these a and its depth h.

If P is a point in this area, the pressure at P is $g\rho h$, and the thrust on the area a is $g\rho ha$.

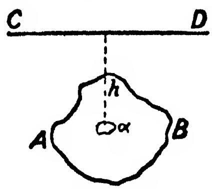


Fig. 105.

Let the plane AB intersect the surface in the line CD. The moment of the thrust on the area round P about this line is $g\rho ha \, . \, h$. The sum of the moments of all the thrusts on all the areas like that round P is $\Sigma g\rho h^2 a$, *i.e.* $g\rho\Sigma h^2 a$. This must be equal to the moment of the resultant thrust about CD. Let the depth of the centre of pressure be h'. Then $h' \times$ total thrust

$$= g\rho\Sigma h^2 a,$$

$$\therefore h' = \frac{g\rho\Sigma h^2 a}{g\rho\Sigma ha} = \frac{\Sigma h^2 a}{\Sigma ha}$$

$$= \frac{\int h^2 dS}{\int h dS} = \frac{\text{moment of inertia of area round intersection of plane with surface}}{\text{moment of area}}$$

220. Centre of Pressure in particular cases.

(1) *Parallelogram* with one side in the surface. $h = \frac{2}{3}$ vertical breadth.

(2) *Triangle* with one side in the surface. The centre of pressure is half way down the median.

(3) *Triangle* with its base parallel to the surface and its vertex in the surface. The centre pressure is three-quarters way down the median.

(4) *Circular disc.* The depth of the centre of pressure below the centre of the circle $= \frac{r^2}{4d}$ when r is the radius and d the depth of the centre.

All these results can be obtained from the relation

$$h' = \frac{\int h^2 dS}{\int h dS}.$$

They may also be obtained without the use of the calculus. (See Greaves' *Hydrostatics.*) In all of them the surface is supposed to be free, *i.e.* not exposed to external pressure.

The results are also true if the planes are not vertical, but in such cases height or depth must be taken to mean the slant distance from the surface measured in the plane of the figure.

221. Transmissibility of Fluid Pressure.—If in a fluid at rest an external thrust be applied at any place causing the pressure at any point to change, then an exactly equal change in pressure is brought about at every other point in the fluid. The principle is known as *Pascal's Theorem,* or the principle of the transmissibility of fluid pressure.

222. The Siphon.—A siphon is a bent tube used for emptying vessels which it is inconvenient to tilt.

Suppose V is a vessel containing a liquid of density ρ.

Let ABC be a siphon filled with the same liquid.

Suppose the height of B above the liquid is h_1, that of B above C is h_2. Imagine the end C to be closed with a cork. We shall find the difference in pressure on the two sides of the cork.

The pressure at the surface of the liquid is atmospheric $(= P)$

pressure at $B = P - h_1\rho$,

pressure at C inside the tube $= P - h_1\rho + h_2\rho$.

Hence the pressure at one side of the cork would be $P + \rho(h_2 - h_1)$, and the pressure at the other side would be P.

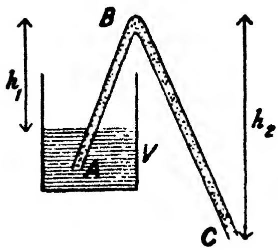


Fig. 106.

If $h_2 > h_1$ there is a greater pressure outwards from the tube on the cork than inwards. There would then be a tendency for the cork to be thrust outwards and for the liquid to follow.

Hence if a siphon works the lower end of the tube must be below the level of the liquid in the reservoir.

Compare the motion of the fluid with that of a rope coiled on a table passing over a pulley with the other end resting on the floor. The longer piece will pull the other over so that the rope will pass from table to floor. The rope of liquid is kept from parting by the pressure of the atmosphere.

The pressure at B is $P - h_1\rho$; this cannot be less than zero: hence $h_1 < \frac{P}{\rho}$. The greatest height of a siphon is, therefore, that of a barometer filled with the fluid. Thus if the water barometer stands at 30 feet, water cannot possibly pass a height of more than 30 feet.

223. Velocity of Efflux: Torricelli's Theorem.—Suppose a mobile liquid (s.g. $= \rho$) is contained in a wide tank and that a small hole is made in the side. The liquid will squirt out and fall in a parabola. The problem is to find the velocity of efflux. Let us suppose the motion is steady: for this to be the case imagine a steady supply of liquid enters at the top of the vessel so that the height (h) of the free surface above the hole is always the same.

Let a = sectional area of the jet supposed constant near the orifice,

v = velocity of efflux.

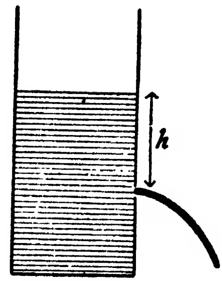


Fig. 107.

Then every second a volume, va, of liquid leaves the tank.

The kinetic energy of this

$$= \tfrac{1}{2} \cdot \rho va. \; v^2.$$

This energy is supplied by the fall of the liquid through a height h.

The work done per second by gravity $= \rho \,.\, va \,.\, hg$.

Hence since work done is equal to energy acquired

$$\rho vahg = \tfrac{1}{2}\rho vav^2$$

i.e.

$$v = \sqrt{2gh}.$$

This result is known as Torricelli's Theorem.

The volume issuing per second is $a\sqrt{2gh}$.

224. Vena Contracta.—The liquid issuing from the hole begins to trace a parabola. By measurement of the parabola it is possible to test the truth of Torricelli's Theorem.

The simplest plan is perhaps to direct the jet vertically upwards: then if the theorem holds the jet should rise to a height, h—*i.e.* up to the level of the liquid in the tank—for that is the height which a body projected vertically with velocity $\sqrt{2gh}$ attains. The actual height reached is slightly less owing to the resistance of the air.

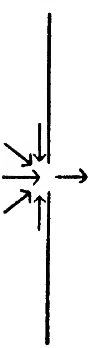


Fig. 108.

The theorem also indicates that the volume issuing per second is av, *i.e.* $a\sqrt{2gh}$.

Experiment shows that this result is much too great. The deviation cannot be sufficiently explained by friction—internal or external. The real reason appears to be that the liquid on entering the orifice is not all moving perpendicular to the side of the vessel (Fig. 108), but has different directions. This obliquity of the motion causes the jet to contract to a neck after leaving the orifice. This neck is called the vena contracta. At this place the

motion at all parts is nearly parallel to the axis of the jet. The actual area of the vena contracta is about ·6 times that of the orifice in the case of a small jet in a thin walled tank, so that Torricelli's Theorem indicates that the volume of liquid issuing per second is $\cdot 6a\sqrt{2gh}$.

The ratio of the vena contracta to that of the orifice is called the coefficient of contraction; it may be found experimentally by weighing the liquid which issues in any interval of time.

EXAMPLES XII.

1. Water flows from a hole (effective area, ·1 sq. in.) in the side of a reservoir. If the hole is 12 feet below the surface, find how many cubic feet of water will escape in an hour.

2. A cubical tank of 1 metre edge is half filled with water and half with oil (s.g. = ·8). Find the thrust on (1) the upper half, (2) the lower half of one of the vertical sides.

3. What are the conditions for the stable equilibrium of a floating body?

A cylindrical cork floats with its axis vertical. If its axis is 5 cm. long, what is the least possible diameter? (S.g. = ·25.)

4. A hollow sphere, radius 10 cm., is filled up to a height of 15 cm. with mercury. Find the resultant thrust on a horizontal surface through the centre of the sphere.

5. A hollow cone of semi-vertical angle 30° and height 20 cm. is filled with water. Find the thrust on the base when the axis of the cone is (1) vertical, (2) horizontal.

6. A siphon is filled with water and inverted into a vessel of liquid of specific gravity 1·6. What is the condition that the liquid may flow through the siphon?

7. Distinguish between the whole pressure and the resultant pressure on a solid immersed in water.

8. A cubic foot of water weighs 1000 oz. A man weighing 160 lbs. floats with 4 cubic inches of his body above the surface. What is his volume in cubic feet?

9. Explain the hydrostatic principle known as that of Archimedes. The specific gravity of gold is 19·3; that of silver is 10·4. What is the composition of an alloy of gold and silver whose specific gravity is 17·6, no change of volume being supposed to accompany the admixture of the metals?

10. A sphere, radius 10 cm., is just wholly immersed in water. Find (1) the whole pressure, (2) the resultant thrust on the surface.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER XIII.

LIQUIDS.

225. Molecular structure of a Liquid.—In Chapter XI. we dealt with gases and imagined them composed of many particles very small, far removed from one another and in rapid motion. The effect of cooling is to diminish the motion, the effect of increasing the pressure is to diminish the distance between the particles. Now cooling and pressure produce a visible effect in many cases—either separately or together they change matter from a gaseous to a liquid state. It is therefore not unreasonable to regard a liquid as being composed of many particles in motion and separated far from one another, but this motion must generally be less vigorous and the free paths shorter than in gases.

The kinetic theory of liquids is perhaps not of the same importance as the kinetic theory of gases, nor is it as simple. Still it is useful to imagine an atomic (or molecular) structure. We picture the process of evaporation as the escape of the molecules through the surface of the liquid. It will be only those molecules which have rapid motion which are shot through the enclosing skin. The slower molecules are left behind. This motion explains the cold produced by evaporation. If evaporation takes place into a closed space, then after a time the space becomes saturated: this means that as many molecules are shot into the liquid from the vapour as pass in the contrary direction.

226. Internal Forces.—To account for the fact that gases do not strictly obey Boyle's law, we assumed that the particles of a gas attracted (or repelled) one another; an assumption justified by the Porous Plug experiment. These forces become greater and more appreciable as a gas is compressed. We may expect then that internal forces act in a liquid, which are much more intense than in the case of gases. These forces are not very readily detected, but their existence is indicated by the surface tension of liquids. In Chapter XV. we shall briefly explain Laplace's theory of capillarity. According to this theory the attractive force between the molecules in a liquid is only appreciable when the distance between the molecules is very small. On the surface of a liquid there are molecules within range in the inner side and none on the outside. Such molecules would therefore be pulled inwards and produce the same effect as a tight skin—*i.e.* produce surface tension.

In the middle of the liquid a pressure will be produced, but this pressure even if intense would not be easily detected, for it must be remembered that we have no means of measuring the pressure of one portion of a liquid on another: all pressure gauges measure the pressure of a liquid on another substance.

It is not commonly known that a liquid can support a tension without parting. The following experiment, devised by Berthelot, shows however that such is the case. He took a glass tube—sealed one end and drew the other out to a fine nozzle. This he filled almost to the nozzle with water and then sealed the vessel, leaving just a tiny bubble of vapour. By gently warming the tube water was made to expand and the bubble disappeared. The tube was thus quite full of liquid. On cooling no bubble reappeared, as must have happened had the water contracted to its old volume. Hence the water was in a state of tension. It was calculated that this tension must have amounted to 200 atmospheres.

This experiment is of considerable importance. It shows among other things that glass and water adhere together with considerable force.

227. Elasticity and Compressibility.—If matter of any kind is subjected to pressure its volume changes: or in other words matter is compressible. The compressibility of a fluid is generally defined as follows: Suppose a volume V of fluid is subjected to a pressure P. Let the pressure be increased by a small amount p: the volume will diminish by some small amount, v say. Then the ratio $\frac{v}{V}$ is termed the compression.

Hence compression

$$= \frac{\text{decrease in volume}}{\text{original volume}}$$

$$= \text{decrease in volume per unit volume.}$$

The ratio of the compression to the extra pressure producing it is the compressibility. Hence in the case considered the compressibility

$$= \frac{\frac{v}{V}}{p} = \frac{v}{Vp}.$$

Volume elasticity is the reciprocal of compressibility.

Hence elasticity $= \frac{Vp}{v} = E.$

It must be noticed that the quantities just defined are not constants: they depend on the original pressure (P) and volume (V).

228. Elasticity of Gases.—In the case of a gas which obeys Boyle's law, we have

$$PV = \text{constant} = (P + p)\ (V - v),$$

$$\therefore\ PV = PV - Pv + pV - vp,$$

$$Pv = pV,$$

since vp, the product of two small quantities, is negligible,

$$\therefore\ E = \frac{Vp}{v} = P,$$

i.e. if a gas obeys Boyle's law its elasticity is equal to its pressure.

If a gas is subject to adiabatic changes, then its behaviour is given by the relation

$$PV^{\gamma} = \text{constant},$$

where $\gamma = \dfrac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}} = \text{constant}.$

In this case then

$$PV^{\gamma} = (P+p)(V-v)^{\gamma},$$

$$\therefore\ 1 = \left(\frac{P+p}{P}\right)\left(\frac{V-v}{V}\right)^{\gamma}$$

$$= \left(1+\frac{p}{P}\right)\left(1-\frac{v}{V}\right)^{\gamma}$$

$$= \left(1+\frac{p}{P}\right)\left(1-\gamma\frac{v}{V}+\text{negligible terms}\right)$$

$$= 1+\frac{p}{P}-\gamma\frac{v}{V},$$

$$\therefore\ \frac{p}{P} = \gamma\frac{v}{V},$$

$$\therefore\ E = \frac{Vp}{v} = \gamma P.$$

Aliter, (1) $$pv = \text{constant},$$

$$\therefore\ \log p + \log v = \text{constant}$$

Differentiate and we get

$$\frac{1}{p}\,dp + \frac{1}{v}\,dv = 0.$$

$$\therefore\ E = \frac{vdp}{-dv} = p.$$

(2) $$pv^{\gamma} = \text{constant},$$

$$\therefore\ \log p + \gamma\log v = \text{constant},$$

$$\therefore\ \frac{1}{p}\,dp + \frac{\gamma}{v}\,dv = 0,$$

$$\therefore\ E = \frac{vdp}{-dv} = \gamma p$$

229. Compressibility of Water.—The experiment described in Art. 226 indicates that water is elastic, for it can be stretched by applying force in a suitable way. To determine the elasticity of water is, however, a difficult problem. Experiments were made at Florence about 1660 with a view to test whether water was at all compressible. Globes made of silver were filled with water and subsequently deformed, but instead of this compressing the water it forced the water to exude through the pores of the metal. This negative result might have been taken to show that liquids are absolutely incompressible were it not that they are known to be media capable of conveying sound with finite velocity. The known velocity of sound in water enables us to form an estimate of the elasticity of water.

We have the relation given by Newton,

$$V=\sqrt{\frac{E}{D}},$$

where $V=$ velocity of sound in a medium,
$E=$ elasticity of the medium,
$D=$ density.

In the case of water, the velocity of sound at 8° C. was found by Colladon to be about 1430 metres per second.

Hence $E=V^2D$

$$=(1{\cdot}43\times10^5)^2\times 1$$

$$=2{\cdot}04\times10^{10}.$$

Compressibility $=\frac{1}{E}=\cdot49\times10^{-10}$.

This is expressed in dynes per sq. cm.

To express it in atmospheres we must multiply by

$$981\times13{\cdot}6\times76.$$

The result, ·000050, is not discordant with that obtained by other methods.

The first successful attempt to show compression in water was made by John Canton, 1762. His method was to take a vessel like a thermometer with a large

bulb and stem of fine bore. This was partly filled with mercury and then warmed till the mercury expanded to fill the whole when the stem was sealed off. After the instrument was cooled the mercury sank some distance on the stem. The mercury was thus subject to its own pressure only. When the extreme end of the tube was broken off, the pressure of the atmosphere caused the mercury to retreat up the stem. This indicated that either the mercury contracted under the extra pressure or that the volume of the bulb increased when pressure was admitted into the interior. If the experiment was carried out with water instead of mercury, the change of the length of the column in the stem was different. The effects were therefore not entirely due to expansion in the volume of the bulb. Canton's attempts to evaluate the compressibility were less successful, as he failed to perceive the nature of the contraction of the glass.

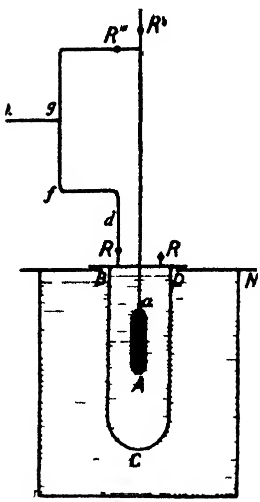


Fig 109.

After Canton the problem was attacked with some success by Orsted, by Colladon and Sturm, and by Regnault. We can here only briefly describe the work of Regnault. His apparatus—a piezometer—is shown in the form adopted by Grassi in Fig. 109.

It consisted of a case *C* with strong glass sides. In this was a vessel *A* with a fine stem *ab*. *A* was filled with the liquid and its height in the stem was measured by a mercury index. The pipe *hg* was connected to an air pump so that any desired pressure could be applied. The following is the order in which operations were conducted:—

1. The taps *R*, *R'''* shut; *R' R''* open. The liquids inside and outside were then at atmospheric pressure.

2. R' was then shut and R opened: pressure is now applied on the outside only.
3. R', R'' shut; R, R''' open: pressure is exerted on outside and inside of the vessel.
4. Close R, open R': pressure on the inside only.

After each of these operations the reading of the index in the tube was taken.

The calculation of the result was dependent on the shape of the vessel A. When pressure was applied equally inside and out, the quantity measured is the difference of compressibility of liquid and glass.

230. Viscosity.—In the treatment of gases in Chapter XI. we have already considered viscosity and defined the coefficient of viscosity for fluids of all kinds. It will be remembered that in the case of a fluid flowing past a solid we assumed that any slipping that took place was in the fluid itself, that there was no slip between the fluid and the solid. Thus in the case of a river the water in immediate contact with the bank or bottom is at rest. The water near to the banks has a smaller velocity than in mid stream. Were this assumption not true the wear of the banks would probably be much greater than it actually is.

231. Poiseuille's method of measuring viscosity.—We shall now investigate the behaviour of a liquid flowing through a long straight tube. We shall find that the quantity of water V which flows out per second is given by the relation

$$V = \frac{\pi p a^4}{8\eta l},$$

where p = pressure difference between the ends of the tube,

a = radius of the tube,

l = length of the tube,

η = coefficient of viscosity.

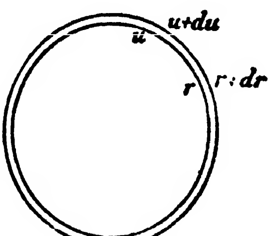


Fig. 110.

The problem was first solved by Poiseuille, who used it to determine the viscosity of fluids.

Consider the motion of a thin shell of the fluid contained between cylinders of radii r, $r + dr$ (Fig. 109). Call the velocities of the fluid on these surfaces u, $u + du$. In the case considered du is negative. Now referring back to Art. 187, where viscosity is defined, we see that the drag back on the inner cylinder per unit area

$$= \eta \frac{(u + du) - u}{(r + dr) - r}$$

$$= \eta \frac{du}{dr}.$$

Now the area of the cylinder $= 2\pi rl$, for it is of length l, and radius r.

$$\therefore \text{ total drag back} = -2\pi l\eta \,.\, r\frac{du}{dr} = F, \text{ say.}$$

Now if the motion is steady, the cylinder radius r of fluid neither gains nor loses momentum.

$\therefore$ $F =$ difference of thrusts between the ends of the cylinder.

If the pressure difference is p, then

$$F = \pi r^2 p,$$

$$\therefore \; -2\pi l\eta r\frac{du}{dr} = \pi r^2 p,$$

$$\therefore -l\eta\frac{du}{dr} = p\frac{r}{2},$$

$$\therefore -l\eta du = \frac{pr}{2}\,dr.$$

Integrate; $$-l\eta u = \frac{pr^2}{4} + k.$$

Now the velocity of the fluid in contact with the tube is zero, since we suppose no slipping to take place,

i.e. $$u = 0 \text{ when } r = a$$

$$\therefore k = -p\frac{a^2}{4}.$$

Hence $$u = \frac{p}{4l\eta}(a^2 - r^2).$$

This equation gives us the velocity of the fluid at any point in the tube.

To find the quantity of the fluid which passes through the tube per second we again consider the thin shell. The velocity of the liquid in this shell is

$$\frac{p}{4l\eta}(a^2-r^2).$$

The cross-sectional area is $2\pi r\,dr$.

$$\text{Hence total volume issuing} = \int^{a}\frac{p}{4l\eta}(a^2-r^2)\,2\pi r dr$$

$$=\frac{p\pi}{2l\eta}\int_0^a (a^2r-r^3)\,dr$$

$$=\frac{p\pi}{2l\eta}\left[\frac{a^2r^2}{2}-\frac{r^4}{4}\right]_o^a$$

$$=\frac{p\pi}{2l\eta}\cdot\frac{a^4}{4}$$

$$\therefore V=\frac{\pi p\,a^4}{8\eta l}.$$

The above investigation is not quite complete. We have omitted to take any account of the kinetic energy of the issuing liquid. In the case of long tubes of fine bore, however, the results are practically true.

232. To find the Viscosity of Water.—A simple apparatus for the determination of the value of η for water is shown in Figure 111. AB is the capillary tube. Tube D is adjustable in position, and when once fixed keeps the pressure head constant during the experiment. With this apparatus experiments can easily be made with various heads, and tubes of different lengths and radii.

233. Temperature Effects.—The viscosity of a liquid decreases as the temperature rises. In the case of gases the viscosity increases as the temperature rises.

234. Diffusion.—If a strong solution of any salt is placed at the bottom of a deep vessel which is then filled up carefully with water, the solution does not remain at the bottom but gradually works up and spreads through the whole vessel. The process of mixing is well shown when the solution is highly coloured. For instance, a solution of permanganate of potash or chromic acid displays the phenomenon well. The process is called diffusion. We have already referred to the work of Graham on the diffusion of gases. He did much work on the diffusion of liquids, and discovered many facts and relations.

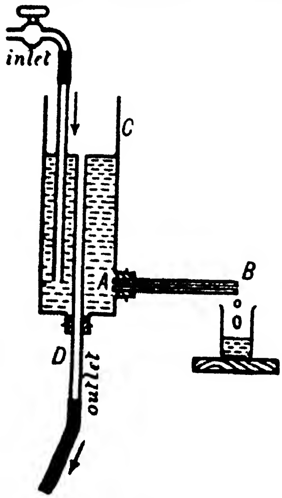


Fig. 111.

235. Graham's Experiments.—One of his methods was briefly as follows: He took a wide-necked bottle filled with the solution on which he wished to experiment and placed it in a larger vessel. This was filled with water above the level of the top of the open bottle. After the lapse of some days the solution was examined to find out how much of the salt had diffused away out of the bottle into the larger vessel.

By use of this simple arrangement he found out that

(1) Solutions of different salts diffuse at different rates.

(2) Solutions may be divided into two classes, crystalloids and colloids. The former, embracing salts, sugar, mineral acids, etc., diffused very much faster than colloids. Colloids include substances such as albumen, glue, and gelatine.

(3) The quantity of a solute (*i.e.* substance dissolved) which passes in unit time from one layer to the next is proportional to the difference in strength between the layers.

(4) The rate of diffusion increases with the temperature.

236. Methods of estimating Concentration.—Experimental work on diffusion is greatly hampered by the difficulty of estimating the strength of a solution at any given point. To withdraw a portion of the solution by means of a pipette sets up currents and is therefore impossible in accurate work. A simple method has been proposed by Lord Kelvin. This method consists in making a series of little beads of different densities. At the beginning of an experiment all these beads float at the surface of separation of the water and solution. As diffusion goes on some of the beads rise higher than others, so indicating at any time the densities at different places. The method is practically not very efficient, as bubbles of air and crystals of salt always form on the beads. In the case of sugars the concentration is estimated by a saccharimeter, an instrument which measures the angle through which the plane of polarization of light is twisted. This angle is proportional to the strength and thickness of the solution. Other possible methods depend on the absorption of light by coloured solution and the changes in the refractive index.

If two plates of amalgamated zinc are placed in a solution of zinc sulphate which is more concentrated near one than the other, then these plates are at different potentials and the electromotive force between them is proportional to the difference in concentration. This leads to an easy method of finding the rate at which diffusion is taking place. Weber makes use of it to determine the diffusivity and to verify Fick's laws. These will be explained in the next article.

237. Fick's Law.—The laws of diffusion were shown by Fick to be exactly analogous to the laws of conduction of heat through a solid. Conductivity (K) is defined by the relation

$$H = K\frac{\theta_1 - \theta_2}{x}At \text{.....................} (1)$$

where H is the quantity of heat which passes in time t between two parallel plane surfaces each of area A at a distance x from one another when these surfaces are maintained at temperatures θ_1, θ_2.

The ratio $\frac{\theta_1 - \theta_2}{x}$ may be termed the temperature gradient.

Simplify the relation (1) by considering a plane of unit area and taking the time as unity. Then the quantity of heat which passes across a plane of unit area in unit time is equal to the product of conductivity and the temperature gradient.

The temperature gradient may or may not be constant throughout the substance. If it is not constant then to get the flow of heat across a plane we must take the distance, x, very small, so that the gradient really is

$$Lt_{x=0} \frac{\theta_1 - \theta_2}{x} = \frac{d\theta}{ds}$$

if we adopt the notation of the calculus and denote by θ the temperature at a plane distant, s, from some fixed plane.

Hence the flow of heat across unit area in unit time

$$= K \frac{d\theta}{ds}.$$

Turning now to the diffusion of, say, salt through a salt solution, we must replace K, the conductivity, by R, the diffusivity, and the temperature by the concentration, and we get the relation :—

"*The quantity of salt which passes across unit area in unit time is equal to the product of the diffusivity and the concentration gradient.*" The law introduces the word concentration, which is taken to mean the quantity of salt (measured in, say, grammes) in unit volume (one cubic centimetre) of solution.

Consider then two parallel planes at a *small* distance x from one another. Let the concentration at these be c_1, c_2, *i.e.* let the strengths at points in them be equal to the strengths of solutions which contain c_1, c_2 grammes of salt per c.c. Then if R is the diffusivity, the quantity of salt Q which passes in unit time across unit area is given by the relation $Q = R\frac{c_1 - c_2}{x}$.

If we use the notation of the calculus, then

$$Q = R\frac{dc}{ds}$$

$$H = K\frac{\theta_2 - \theta_1}{x} = K\frac{d\theta}{ds}; \qquad Q = R\frac{c_2 - c_1}{x} = R\frac{dc}{ds}.$$

This relation is of great importance. It shows us that any result which holds for conduction of heat necessitates a similar result in diffusion.

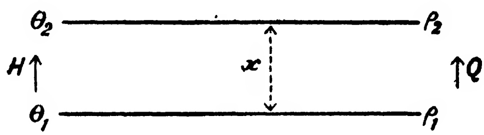


Fig. 112.

238. Relation between Concentration, Time, Distance.

Take two planes at a distance ds apart.

The flow across the first is $R\frac{dc}{ds}$.

The flow across the second is $R\frac{dc}{ds} + \frac{d}{ds}\left(R\frac{dc}{ds}\right)ds$.

The difference between the two is $R\frac{d^2c}{ds^2}\,ds$.

Dividing this by the volume between the planes we get

$$\frac{R\frac{d^2c}{ds^2}ds}{ds} = R\frac{d^2c}{ds^2}.$$

This gives us the rate of change in concentration. But the rate of change in concentration is also $\frac{dc}{dt}$.

$$\therefore\ \frac{dc}{dt} = R\frac{d^2c}{ds^2}.$$

This differential equation is one which must always be satisfied. Solutions of the problem have been found in certain cases.

239. Osmose.—When two liquids which diffuse into one another are initially separated by a membrane, the process of mixing is usually called osmose. Abbé Nollet records an experiment in which he took a jar filled with alcohol. The mouth of this he covered with a bladder. The whole was then immersed in water. The water found its way into the jar faster than the alcohol escaped, with the result that the bladder was made to swell out. In a similar fashion, if a bladder be filled with water and then immersed in alcohol, the water escapes and the bladder shrivels up. Interchange the water and alcohol and the bladder may burst.

240. Dialysis.—Solution of crystalloids can pass through a bladder or parchment membrane much faster than colloids such as gum or albumen. Graham made use of this to separate colloids and crystalloids from one another. The process he termed dialysis. The mixture of crystalloids and colloids is placed in a tray, the bottom of which is made of parchment. This is floated in a vessel of water. The crystalloids pass through the parchment, while most of the colloids remain behind.

This method is employed to detect poisons in the viscera of poisoned animals. It is also used in chemistry to separate salt from silicic acid when the latter is formed by the action of hydrochloric acid on sodium silicate.

241. Osmotic Pressure.—A membrane of copper ferrocyanide, supported by stronger material, is what is usually employed in experimental work. This is formed by placing a porous pot filled with a solution of copper sulphate in a vessel containing a solution of potassium ferrocyanide. The two solutions meet in the walls of the pot, and the precipitate formed makes a membrane which is firmly supported. The pot is then well washed.

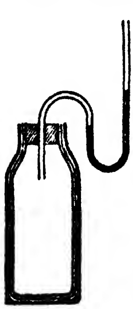


Fig. 113.

Fig. 113 represents such a vessel. The mouth is firmly closed. The pot is filled with a solution and sealed up. The pressure of the solution is indicated by a mercury gauge. If the pot be left exposed to the air, some of

the contents will gradually exude and the pressure indicated by the gauge will fall. If, however, the pot is placed in water, the water will gradually find its way in, and the mercury will rise in the gauge till a certain definite pressure is reached. There is then equilibrium, *i.e.* water passes out at just the rate at which it enters.

The amount of the final pressure is dependent on the nature of the solution, its strength, and its temperature. It is termed the osmotic pressure.

The student may, if he likes, imagine this pressure to be due to bombardments of the sides of the vessel by particles of the solute, that the water can enter or leave freely, but that the membrane acts as a sieve through which the particles of solute cannot pass.

242. Laws relating to Osmotic Pressure.—It is very remarkable that dilute solutions appear to obey laws exactly analogous to the laws of gases.

(1) Boyle's Law in a modified form holds for solutions, for the osmotic pressure of a dilute solution is directly proportional to its concentration, *i.e.* to the amount of solute per c.c.

(Pressure $\propto$ Density.)

Some of the experimental results of Pfeffer are as follows:

(1) Percentage of sugar..	1	2	4	6	
(2) Osmotic pressure.....	535	1016	2082	3075	mm. of mercury.
Ratio of (2) : (1) ...	535	508	521	513	

(2) The osmotic pressure of a solution increases with rise in temperature. No very complete experiments in this direction seem to have been made, but it would appear that the pressure is nearly proportional to the absolute temperature. (Charles' Law, $P \propto T$.)

(3) Solutions of non-electrolytes which have equal osmotic pressures contain the same number of gramme-molecules per c.c., or in other words are of concentrations proportional to the molecular weights of the solute. Compare this with Avogadro's Law, which states that gases at

the same pressure contain the same number of molecules per c.c. Solutions exerting the same osmotic pressure are termed *isotonic*.

(4) The osmotic pressure of the solution of a non-electrolyte is equal to the pressure that would be exerted by the solute were it capable of existing as a gas at the temperature and volume of the solution. Thus Pfeffer found that at 6·8° C. a 1 per cent. solution of sugar had an osmotic pressure of 505 mm. of mercury. The molecular weight of sugar ($C_{12}\ H_{22}\ O_{11}$) is 342. Now if a gas had a molecular weight of 342, its volume at a pressure of 505 and temperature 6·8° C. would be $\frac{2 \times 11160}{342} \times \frac{760}{505} \times \frac{279{\cdot}8}{273}$ c.c.,

i.e. 100·7 c.c.

(5) Solutions of electrolytes exert a greater osmotic pressure than solutions of non-electrolytes. This is probably due to the fact that electrolytes in solution split up into ions. Compare with the (so-called) abnormal vapour densities of substances such as calomel and sal ammoniac.

243. Isotonic Solutions.—Direct experiments in osmotic pressure are cumbrous and not susceptible of great accuracy. Hence many results have been obtained by somewhat indirect methods. De Vries experimented on the cells of certain plants. These cells are lined with a semi-permeable membrane through which water can pass, but which does not allow the passage of solutions. Hence if such a cell is placed in water, the water enters—as in the case of Nollet's bottle—and the membrane presses up against the walls of the cell: but if placed in a strong solution of salt, water passes out and the membrane shrinks away from the walls.

The strength of a solution can therefore be varied till the cell neither shrinks nor swells. The solution is then isotonic with the contents of the cell, *i.e.* the two exert equal osmotic pressures. De Vries used these cells to find out what strength solutions of different substances must have to be isotonic with one another. It was by this means that Law (3) of Art. 242 was confirmed, and also the molecular weight of raffinose.

244. Vapour Pressures of Solutions.—It is well known that the boiling point of water is lower than that of a salt solution; as liquids boil when their vapour pressures are equal to the pressure of the atmosphere, it follows that the vapour pressure of a salt solution is lower than that of pure water. This lowering of the vapour pressure is connected with the osmotic pressure.

Imagine a tube AB (Fig. 114) closed at the bottom with a permeable membrane M. Let this be filled with a solution and placed with the bottom just touching the surface SF of water in another vessel. It will simplify matters if we suppose the whole to be placed in the exhausted receiver of an air pump.

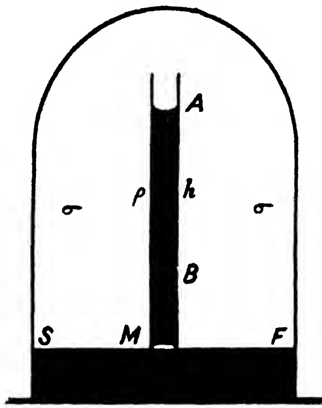


Fig. 114.

If the solution in AB is strong and the length of the column of solution is not too great, water will pass through M into the tube, and the solution will climb up higher and higher till the hydrostatic pressure at the bottom of the tube is equal to the osmotic pressure of the solution in its then state of dilution. There will then be equilibrium.

Now the whole of the space under the receiver is filled with water vapour. This vapour is saturated throughout, both above the solution and above the water surface. But the vapour pressure at A, *i.e.* the saturation vapour pressure of the solution, is less than the vapour pressure of pure water just above SF by an amount equal to the weight of a column of vapour on base of unit area and height equal to the difference in level between the free surfaces. Hence vapour pressure of water (p_w)

$$- \text{vapour pressure of solution } (p_s) = g\sigma h \text{............(1)}$$

where σ denotes the average density of the vapour and h the difference in level.

The osmotic pressure P = hydrostatic pressure at M in the solution,

$$i.e. \qquad P = g\rho h \quad \text{..........................} (2)$$

when ρ is the density of the solution.

Divide (1) by (2) and we get

$$\frac{\text{Vapour pressure of water—Vapour pressure of solution}}{\text{Osmotic pressure}}$$

$$= \frac{p_w - p_s}{P} = \frac{\sigma}{\rho}.$$

245. Lowering of the Vapour Pressure.—Now the vapour pressure of a solution is not a very easy thing to measure. As a rule it is simpler to measure the difference in temperature between the boiling points of solution and solvent.

If the solution is dilute, then the fraction

$$\frac{\text{Difference in pressure between vapours of solvent and solution}}{\text{Difference in boiling point}}$$

is almost equal to the rate at which the pressure of the vapour of the pure solvent rises with the temperature, *i.e.*

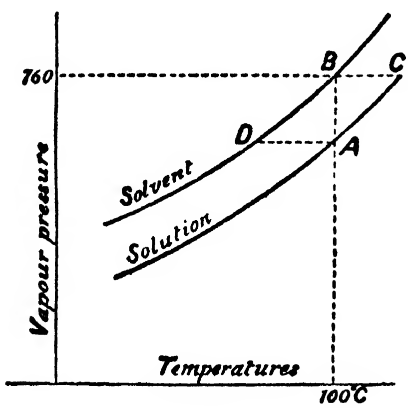


Fig. 115.

to the rise in pressure per degree rise in temperature. Call this w and the difference in boiling points τ,

then $$\frac{p_w - p_s}{\quad} = w.$$

Thus in Fig. 115 the upper curve shows the relation between the vapour pressure of a solvent (water in this case) and its temperature: the lower curve exhibits the relation for a solution.

Now at 100° C. the difference in the two vapour pressures is represented by AB, the difference in boiling point by BC. The fraction $\frac{AB}{BC}$ is nearly equal to $\frac{AB}{AD}$, *i.e.* to the tangent of the angle which the tangent to the upper curve makes with the line of temperatures. The tangent of this angle measures the rate at which the vapour pressure of the solvent varies with the temperature.

246. Lowering of Freezing Point.—A relation exists between the osmotic pressure of a solution and the difference in temperature between the freezing points of the solution and the solvent. Take a simple case: let a dilute solution of salt at its freezing point, $273 - t$ absolute, be separated by a semi-permeable membrane from the solvent, water, at its freezing point 273.

Let one gramme of water pass through the membrane in the solution: the work done is P, when P is the osmotic pressure of the solution. Freeze out the water that has passed through, place it in the pure water and melt it. The heat supplied is that required for melting one gramme of ice: call this L. The system has not been in any way altered by this cycle of operations: also the cycle is reversible, so that

$$\frac{\text{Work done}}{\text{Heat supplied}} = \frac{\text{Difference in temperature}}{\text{High temperature}}.$$

i.e.
$$\frac{P}{L} = \frac{t}{273}.$$

Hence
$$P = \frac{L}{273}t,$$

i.e. the lowering of freezing point is proportional to the osmotic pressure.

Examples XIII.

1. Define vapour tension, vapour density, surface tension, and viscosity. How do all or any of these affect the draining and drying of a long vertical tube open at both ends and wetted inside with the liquid? If any other physical properties of the liquid affect the draining or the drying, mention them. Consider cold water, water near the boiling point, and cold ether or benzine in your answer.

2. Describe the phenomenon of tensile strength in liquids. How has it been measured?

3. Show how the viscosity of a liquid can be calculated from its rate of flow through a narrow tube.

4. Use the method of dimensional equations to show that the terminal velocity of a drop of rain varies as the square of the diameter of the drop.

[Lamb, *Hydrodynamics*, gives the formula $u = \frac{2}{9}\frac{\rho_0 - \rho}{\eta} ga^2$ for a sphere of radius a, density ρ_0, falling through a medium of density ρ. The relation is only true for small drops.]

5. A ball of iron, of density 7, is allowed to fall freely through water. Neglecting the mass of any water carried down by the ball, find the initial acceleration, and explain in general terms why the acceleration changes (1) with the time of fall, and (2) with the size of the ball.

6. Trace the connection between the osmotic pressure of a solution and its solvent. Describe any experiments you know on the vapour pressures of solutions and discuss the general results obtained.

7. If P is the osmotic pressure of a solution, D the density, L the latent heat of the solvent, prove that the freezing point of the solution is lower than that of the solvent by an amount $\frac{PT}{LD}$, where T is the freezing point (absolute) of the solvent.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER XIV.

FRICTION.

247. If a book is lying on a table it is necessary to apply force of a certain definite amount before it can be made to slide along the surface. A force of less magnitude will produce no visible effect. It will only deform the book and bend the table to an inappreciable extent. The smoother the table and the lighter the book the less is the minimum force that must be applied to produce motion. Further, when the book has been set in motion its acceleration is not equal to the force applied divided by the mass: it is considerably less. These facts indicate that resistance of some sort is offered to the motion of one body over another with which it is in contact. The resistance is called friction.

248. Statical Friction.—Consider two bodies in contact with one another. Let forces act tending to make one move over the surface of the other. Let these forces be very small at first and then gradually increase till slipping is just about to take place. Frictional forces act during the whole period. They obey certain important laws:—

Law I.—The direction and magnitude of frictional forces are such as just to preserve equilibrium; provided that at no point the amount necessary exceeds a certain amount called Limiting Friction.

249. Limiting Friction can be investigated easily by the arrangement shown in Fig. 116.

In this AB is a horizontal slide of wood, iron, or other material. C is a sledge that can be drawn over the slide by the string DE, which passes over the pulley P and supports a scale pan Q. C may be loaded as required.

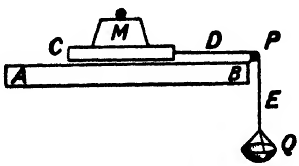


Fig. 116.

To experiment, place a load M on the sledge: add weights to the scale pan Q till motion is just on the point of taking place. The friction which prevents motion is now limiting friction. Its magnitude is practically equal to weight of the scale pan and its load.

The forces acting on the loaded sledge are:—

(1) Its weight, W, vertically down.

(2) The normal reaction, R, of the slide. This is equal and opposite to W.

(3) The pull of the string, T, horizontal.

(4) The friction, F, equal and opposite to T.

Now if experiments are made with different loads, it will be found that $W : T$ is a constant. Hence F is proportional to R; or $F = \mu R$. Here μ is constant; it is termed the *Coefficient of Limiting Friction*. Its value differs considerably in different materials, though it would appear to be always less than unity. If the sledge C is replaced by another of the same material but of different area, the same relations will still hold: the friction is independent of the areas in contact.

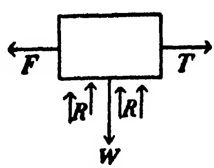


Fig. 117.

250. We may now state the next two laws of friction:—

Law II.—The limiting friction between two surfaces is proportional to the normal reaction between them. From this it follows that

Law III.—Friction is independent of the area of the surfaces in contact.

251. It will be noticed that when motion once takes place the sledge has a considerable acceleration. This points to the fact that resistance to motion is less just after slipping than it was before. Experiments have been made which indicate that the friction between moving bodies is independent of their relative velocity. This point will be discussed later.

Law IV.—Where motion takes place the frictional resistance is less than limiting friction, and is (nearly) independent of velocity and area of surfaces in contact.

252. Angle of Friction.—Refer back to Fig. 117. The forces R, F may be compounded by the parallelogram law: their resultant makes an angle, θ say, with the direction of R, such that its tangent is equal to $\frac{F}{R}$,

i.e. $$\frac{F}{R} = \tan\theta.$$

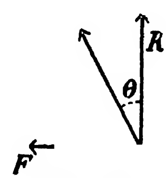


Fig. 118.

If motion is just on the point of taking place, $\frac{F}{R} = \mu$.

In this case the angle θ is generally denoted by λ, and is termed the angle of friction.

$$\therefore \mu = \tan\lambda.$$

Now if motion is not on the point of taking place the friction F must be less than limiting friction:

i.e. $$F < \mu R,$$

$$\therefore \tan\theta < \mu,$$

$$\therefore \tan\theta < \tan\lambda,$$

or $$\theta < \lambda,$$

so that the total reaction makes with the normal an angle (θ) which is never greater than the angle of friction (λ).

253. Rough Inclined Plane.—Suppose a body rests on a plane inclined at an angle θ to the horizon The forces acting on it are

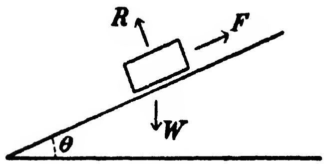


Fig. 119.

(1) Its weight W, acting vertically down.

(2) The normal reaction, R, at right angles to the plane.

(3) The friction F, acting in the direction opposite to that in which the body tends to move, *i.e.* up the slope of the plane.

Resolve horizontally and we get

$$F \cos \theta = R \sin \theta,$$

i.e.
$$\frac{F}{R} = \tan \theta.$$

Now let the plane be tilted up until the body is on the point of slipping. In this case

$$\tan \theta = \frac{F}{R} = \mu = \tan \lambda,$$

i.e. the body is on the point of slipping when the plane is inclined to the horizon at the angle of friction. This method is sometimes a convenient one for finding λ and μ.

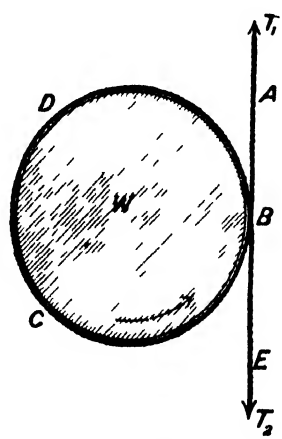


Fig. 120.

254. Work done by Friction: Rope Brake.—Suppose a wheel, W, such as the fly-wheel of a steam engine, rotates at n revolutions per minute. Let a loop of rope $ABCDE$ be passed round it. The friction of this rope will act as a brake on the wheel. The tension of the rope will be greater at E than at A. Call the tension at these two places T_2 and T_1 pounds weight. The effect of the rope is therefore to apply a tangential force $T_2 - T_1$ tending to

stop the wheel, so that the work done by the wheel is exactly the same as it would be were the wheel to draw up a load of weight $(T_2 - T_1)$ hung from a rope fastened on the circumference.

Hence the work done per revolution

$$= (T_2 - T_1) \cdot 2\pi a \text{ foot pounds,}$$

a being the radius of the wheel in feet. (For simplicity we have neglected the effect of the thickness of the rope.)

Hence work per minute $= 2\pi na\,(T_2 - T_1)$.

$\therefore$ horse-power of engine driving the wheel

$$= \frac{2\pi na\,(T_2 - T_1)}{33{,}000}.$$

This is the usual method employed to find the brake horse power of an engine.

It is important to notice that the frictional forces do not depend on the radius of the wheel: the work done against them per revolution is proportional to the radius. Hence we see that the less the diameter of the axle on which a wheel turns the less the losses due to friction.

255. String on a Rough Surface.—Suppose a rope lies on a curved surface and is just on the point of slipping. Let T_1, T_2 be the tensions at the free ends, ψ the angle between the directions.

Then $$T_1 \div T_2 = e^{\mu\psi}.$$

This may be shown as follows. The forces acting on an element $PQ = ds$ are indicated in Figure 121. Resolve along the tangent and normal at P. We get

$$\begin{cases} \mu R ds = dT \\ T d\psi = R ds. \end{cases}$$

$$\therefore \mu T d\psi = dT.$$

$$\therefore \mu d\psi = \frac{dT}{T}$$

$$\therefore \log T = \mu\psi + \text{const.},$$

or $$T_1 = T_2\, e^{\mu\psi}.$$

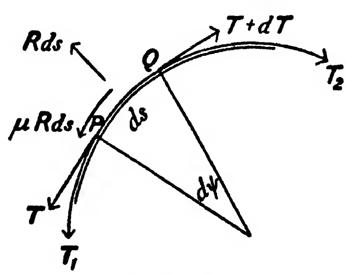


Fig. 121.

The result may be examined by passing a string round a

horizontal cylinder (Fig. 122) and attaching scale pans to the free ends. The string may have $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, etc., laps; so that ψ can be made to have values π, 3π, 5π, etc.

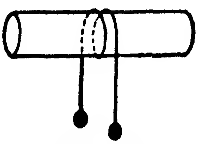


Fig. 122.

256. Change of Friction with Velocity.—The apparatus shown in Figure 123, which is taken from Prof. Perry's *Applied Mechanics*, was designed to show how the friction between the disc D and the brake A changed with the relative velocity of the surfaces in contact. The disc could be driven by an engine at any speed desired:

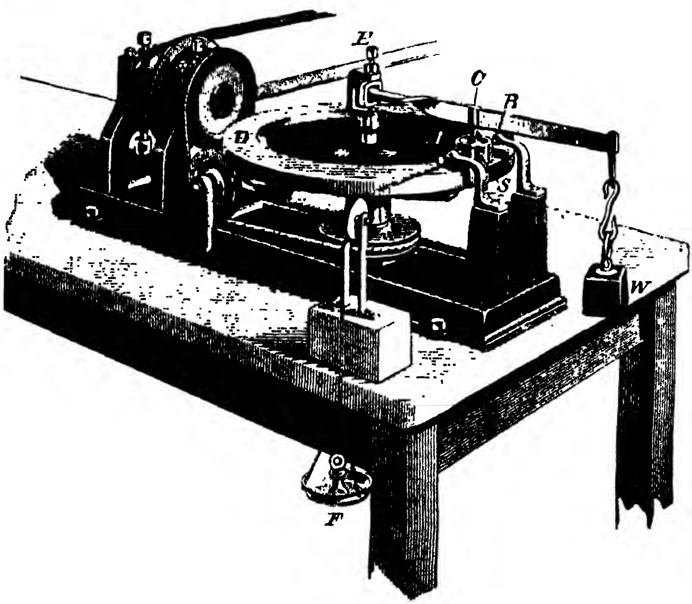


Fig. 123.

the weight supported by the brake could be varied. As the disc revolved in the direction shown by the arrow friction tended to pull the brake round too. It could be prevented from doing so by adding weights in the scale pan F. The results obtained indicated that the friction was not

quite constant but greatest at slow speed. The effect of the wheel, however, is to carry air in under the slide: this would appear to act as a sort of lubricant.

257. Rolling Friction.—When one body rolls over another the resistance called into play is termed rolling friction. This resistance is partly due to the crushing and bending of the material. The stresses and strains which are brought into play as an elastic body rolls along an elastic surface are, needless to say, of a very complicated nature. Under ordinary circumstances there will be temporary alterations in the shape both of the rolling body and of the surface on which it rolls. There may also be a certain definite tendency to adhesion.

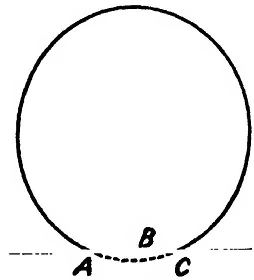


Fig. 124.

Consider the case of a bicycle tyre (Fig. 124). The part ABC in contact with the road is flattened out. The tyre is bulged just in front of C and the road slightly humped up. These strains absorb energy, which is not wholly restored to the wheel by the reaction of the road on the tyre at A. Hence force is required to make a wheel roll along a flat road. The rolling friction is less the greater the diameter of the wheel. In any case it is of much less importance than the sliding friction at the axle of the wheel. Hence the value of ball bearings.

258. Fluid Friction.—This subject has been already studied under the chapters dealing with liquids and gases (see Arts. 185, 230). It must be remembered that the motion of a solid through a liquid is not hampered by frictional forces alone. Thus when a ship moves through water it creates a train of waves at the bow and another at the stern. The energy required to maintain these is supplied by the engines.

Skin resistance is due to the water slipping past the sides of the vessel. It appears to vary approximately as

the square of the speed. Lord Kelvin's *Popular Lectures* contain an interesting account of the resistance offered to the motion of ships.

EXAMPLES XIV.

1. A cube rests with one edge on a rough horizontal plane, and a parallel edge on a smooth plane inclined at 45° to the horizon. If θ is the inclination of the upper face of the cube when just about to slip, prove that $(1-\mu)\cot\theta = (1+3\mu)$.

2. A body will just rest on a plane inclined to the horizon at 20°. Find its acceleration when the plane is inclined at 40°.

3. The fly-wheel of a 20 h.p. engine revolves 250 times per minute. If the brake wheel is 4 ft. diameter, find what the load on the rope brake must be when all the power is being absorbed.

4. A string passes over a rough horizontal cylinder. It carries at its ends masses of 15 lbs., 5 lbs., and is just on the point of slipping. What is the coefficient of friction?

If the string be wrapped round the cylinder for an extra turn, what is the least load that would be required to support the 15 lbs.?

5. A 2 h.p. motor bicycle can go at 15 miles per hour on a level, exerting only 1 h.p. What is the steepest hill it can climb at the same rate when working full power? Weight of bicycle and rider equals 300 lbs.

6. State shortly what you know about the action between two rough surfaces in contact.

A heavy uniform rod lies on a rough horizontal plane and is acted on at one end by a force in the plane at right angles to its length. Show that as the force increases the rod will begin to rotate about a point dividing the rod approximately in the ratio 29 : 70.

(*For further examples on this chapter see Miscellaneous Examples, p. 258.*)

CHAPTER XV.

CAPILLARITY.

259. Behaviour of Small Quantities of Liquids.—If a small quantity of a liquid is placed on a smooth horizontal surface, we might expect it to spread out in a thin film of uniform thickness: this result would seem to be a direct and necessary consequence of gravitation. If experiments are actually carried out the results will be found to differ with the nature of the surface and the liquid. Thus a little drop of paraffin oil allowed to fall on the surface of still water spreads out in a film which may be sufficiently thin to show interference colours. Water on polished glass behaves in a similar way. With mercury on glass, however, we obtain different results, for the mercury, instead of spreading out gathers up into a pool which is circular and several millimetres deep. If it is divided up into separate parts it will be noticed that each is circular in outline and that if two of them come into contact they unite together and again form a circular pool. Instead of glass we may use almost any non-metallic surface—wood, paper, etc.—and obtain similar results. This is not an effect peculiar to mercury; a drop of water will not spread over a greasy plate, but gathers up like mercury on glass.

It is well to notice here that the liquids only spread out indefinitely provided that they are capable of wetting the surface on which they are spread. In other cases the liquids collect in drops or pools. A study of these results suggests that the behaviour of a liquid is not wholly con-

trolled by its weight or forces we have hitherto considered, but that there are other forces in action which are dependent on the natures of the surfaces in contact. These forces are called surface tensions, but before discussing them it will be well to call to mind a few familiar facts and describe some simple experiments.

260. Shape of Liquids Unacted on by External Forces.— A drop of liquid is perfectly spherical provided it is not forced to take up any other shape by external forces. This property is utilised in the manufacture of shot. Molten lead is poured down in a fine stream from the top of a high tower. The stream breaks up into a succession of small drops which cool and solidify as they fall through the air. To prevent them from being knocked out of shape when they reach the ground their fall is broken by a deep bath of water in which they are caught.

The rainbow affords proof that water left to itself takes up the form of a sphere. The explanation of the shape and colouring of rainbows and haloes is based on the assumption that raindrops are spherical. Any slight deviation from this shape would be sufficient to completely change the character of the phenomena.

The effects of gravity on the shape of the liquids in the above illustrations is almost absent because the drops fall freely. If the drops fell through a viscous or dense medium their forms would be altered.

Plateau's Experiments.—Another method of eliminating the effects of gravity is described by Plateau in his "*Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires.*" He prepared first a mixture of alcohol and water of specific gravity as nearly as possible equal to that of oil. This was placed in a glass vessel and a quantity of oil poured in. As the bottom of the mixture was slightly more dense than the oil and the top slightly less so, the oil sank halfway and floated in the middle as a round ball. By means of a disc attached to a wire Plateau made the ball revolve round a vertical axis. The effect of this rotation was to cause a flattening of the poles which increased with the speed. The oil eventually formed a ring, which broke up into separate spheres. This experiment is interesting as suggesting the explanation of the flattening of the earth at the poles and of the stability of Saturn's rings.

261. Liquid Skins: Comparison with Sheet Rubber.— If an elastic bag—a child's balloon or a football bladder—is inflated with air the shape that it takes up is always that of a sphere, provided that the material of which it is composed is uniform in quality and thickness. Here we have two opposing sets of forces to be considered, the one the tensions in the material of the bag, the other the thrusts exerted by the compressed air. In accordance with the former the bag assumes the least possible surface, and in accordance with the latter its volume is as large as possible. Since the shape assumed is spherical we are led to the conclusion that the sphere is the figure which for a given volume has the minimum surface area. This conclusion is capable of rigid mathematical proof.

We may now consider the case of the same elastic bag filled with a heavy liquid and resting on a table. If the liquid is under pressure the forces acting are similar to those indicated above, but a third force, gravity, now comes into operation and produces important effects. The bag is no longer spherical: the portion resting on the table is flat and the curvature of the top decreased: a horizontal section through any point is circular, so that the figure is a solid of revolution.

Its shape is so similar to that of a drop of mercury (Fig. 125) resting on a horizontal plate of glass that it is reasonable to assume as a working hypothesis that the forces which mould the drop are the same as, or at least similar to, the forces regulating the shape of the liquid in

Fig. 125.

the bag. In other words the behaviour of a drop of liquid is explicable if we regard it as being covered with a thin tightly stretched elastic skin. The tension of this skin or film is called the surface tension of the liquid. It is due to the tension of this skin that the hairs of a wetted paint-brush hold together.

262. Formation of Drops.—The following experiments can easily be performed and show the effects of surface tension.

Make an arrangement by which water may be made to flow very slowly from the nozzle of a vertical pipe or tube. The diameter may be anything less than half a centimetre. The flow of the water may be regulated by connecting the pipe to the reservoir by means of rubber tubing, which can be squeezed together by a screw pinch-cock. Observe very carefully the shape of the drops as they are formed. Drops formed in this way are very small, for the skin is not strong enough to bear the weight of the drop when it gets large. We can produce larger drops by overcoming the effects of gravity in the way adopted by Plateau in his experiments. To do this make a solution of zinc sulphate of such strength that its density is just less than that of carbon disulphide. To differentiate the two liquids colour the carbon disulphide with a little iodine. Now take a wide glass cylinder, half fill with the disulphide and pour in the zinc sulphate solution nearly to the top. Take a piece of large bore glass tubing, close one end with the hand, and holding it vertical lower the other end to the bottom of the cylinder. Open the upper end to allow the carbon disulphide to enter the tube and close it up again. Raise the tube till the lower end is well above the surface of separation of the two liquids, and then allow drops to form by admitting air very gradually at the upper end. Notice how a narrow neck is formed just before the drop breaks off. This neck goes to form the little subsidiary drop which falls directly after the main drop is detached.

These effects may be imitated by means of sheet rubber. To do this the rubber must be stretched evenly across a ring or the bottom of a wide pipe and water poured in at the top. The rubber stretches and the appearance is at first almost identical with that of a drop in the early stages of its formation: but when the neck is formed the resemblance ceases; the reason being that whilst in rubber the tension increases with the stretching, in water the tension of the skin always remains the same.

263. Oscillations of a Drop.—A drop just before it leaves the nozzle on which it is formed is not spherical, but is pulled out with the vertical diameter longer than the horizontal. At the moment of separation its outline is something like that of a lemon. While falling freely the influence of gravity on the shape is negligible, so that the surface tension alone need be considered. This would cause the drop to become a perfect sphere, and for an instant the drop has this appearance. The particles, however, of which the drop is composed, have acquired energy and momentum from the action of the surface forces, and they cannot instantly come to rest relative to one another. Hence the drop does not remain spherical, but becomes flattened like an orange. The drop therefore vibrates and passes through different shapes, prolate, spherical, oblate. These vibrations gradually get less and less till they are finally quenched by internal friction.

It is owing to these alterations in shape that a stream of falling drops of water flowing from a tap or jet has a twisted or screwed appearance. Such a stream often seems to be continuous, but this is an optical illusion, due to the persistence of vision. An instantaneous photograph taken by the spark of a Leyden jar shows the stream broken up into its component drops (Fig. 126).

Fig. 126.

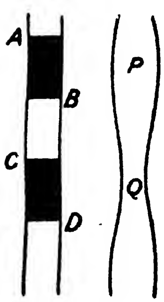


Fig. 127.

A stream of water often has a twisted appearance before it breaks up into drops. This is caused by "oscillations of a cylinder." While the portion AB (Fig. 127) is travelling to CD it is in a state of oscillation. The section at AB may be elliptical with the long axis in the plane of the figure at P and the short at Q. Other kinds of oscillation can be observed.

Lord Kelvin has calculated the period of oscillation of a drop of water through the forms described above and finds that it is about $\frac{1}{4}a^{\frac{3}{2}}$, where a is the radius measured in

centimetres; thus for a drop $\frac{1}{2}$ cm. in diameter the period is $\frac{1}{32}$ second. (*Popular Lectures;* cf. also Chap. I., Ex. 4.)

264. Plane Soap Film.—Make a plane ring of wire; take a thin cotton thread and tie it loosely across it. If the ring is now dipped into a soap solution and removed a thin plane film will be obtained divided irregularly into two parts by the cotton. Break one portion of the film—this can be done successfully by touching it with a hot wire—and the cotton will be pulled out into the form of an arc of a circle.

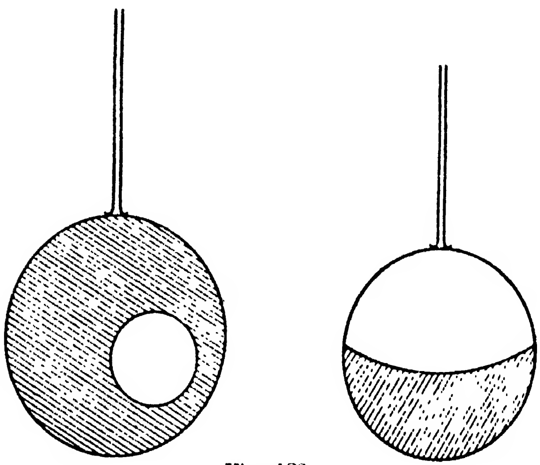


Fig. 128.

Again take a fibre of cotton and tie the ends together. Dip the ring of wire into the soap solution and lay the loop of cotton on the film obtained. Break the film inside the loop. The cotton will be pulled out into the shape of a circle.

A circle is the plane figure which for a given perimeter has the maximum area; the soap film takes the form which has the least possible area; the space unoccupied must therefore be a maximum, and the cotton must therefore be pulled out into a circle.

265. Surface Tensions in different Liquids. Tears in Wine.—Pour into a flat dish just sufficient water to cover the bottom. Let a drop of ether or alcohol fall into the middle. The water will retreat from the ether and heap itself up round the edges of the dish, leaving the ether in the middle. The motion of the water may be rendered more conspicuous by colouring it or sprinkling a little dust—lycopodium powder—over the surface. The reason for this motion is that the surface tension of ether is much less than that of water: the water skin is stronger than the ether skin. Hence the ether surface is stretched out while that of the water contracts. It is well known that if port or other strong wine is standing in a glass the sides of which are wetted with the wine, it gathers itself together in little streams and runs down the glass in tears. The explanation of this is usually given as follows. The surface tension of alcohol is much less than that of water, so that as the alcohol in the wine on the sides of the glass vessel evaporates leaving the less volatile water behind, the surface tension above is greater than that of the unaltered wine below. This causes more wine to be pulled up on the sides, from which the alcohol again evaporates, leaving water which accumulates till a drop is formed sufficiently heavy to break away and fall as a tear.

The motion of camphor on water is due to this cause. If a piece of it is fastened to the stern of a small wooden boat, the boat will move, for the bows are pulled forward with a force due to the surface tension of pure water, while the drag back on the stern is due to a surface contaminated by camphor. As the surface tension of water is greater than that of a camphor solution the boat travels forward.

266. Definition of Surface Tension.—Consider any *small* rectangular element of a surface that is in equilibrium. Call it $ABCD$. The surface forces acting on the edge AB may be resolved into two—one acting on the edge, the tangential force, and one in the surface at right angles to AB. Now if $ABCD$ is the surface of a liquid the former of these components cannot exist. This is due to the fact that a liquid surface, unlike a sheet of paper or piece of cloth, cannot withstand permanently any shearing stress, however

small. Hence the forces acting on the edge of any element taken in the surface of a liquid at rest are everywhere in the plane of the surface at right angles to the edges. The total force acting on the element of the edge divided by the length of the element—or in other words the force per unit length—is termed the surface tension. The dimensions of surface tension are therefore 1 in Mass, – 2 in Time. (Force ÷ Length.)

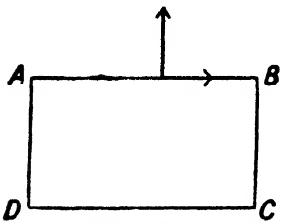


Fig. 129.

267. Surface Energy: $S = T$.—In dealing with capillarity it is often more convenient to consider not the actual surface tensions or the forces acting on the liquid, but the energy associated with the surface. The surface of a liquid tends, as we have seen, to become as small as possible. Now a system is in stable equilibrium when its potential energy is a minimum. Hence we may regard the surface of a liquid as being possessed of potential energy, each square centimetre having a certain amount which is independent of the configuration.

Let us consider a plane liquid film—say a film of soapy water.

Suppose that

T = surface tension of each side of the film,

S = energy associated with each square centimetre.

Let the film be held in a rectangular wire frame $ABCD$, so arranged that the cross wire BC can slide along the sides AB, DC. Let the wire be pulled along to the position $B'C'$ through a distance x. Then if a is the length of BC, the work done in stretching the film $= 2Tax$.

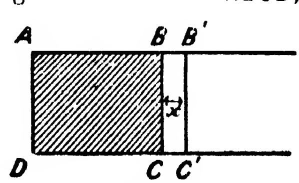


Fig. 130.

The potential energy of the film has therefore been increased by an amount $2Tax$. Each side of the film has

increased by an area ax. Hence increase of energy per unit area $= \frac{2Tax}{2ax} = T$.

If we call S the surface energy per unit area this gives us $S = T$. The dimensions of S are 1 in Mass and -2 in Time. (Work $\div$ Area.)

268. Surface Tension is the same in all directions and at all points.—Consider the equilibrium of the surface of a liquid. Take a small element in the surface in the shape of a right-angled triangle ABC. Let the surface tensions in the edges BC, CA, AB be T_1, T_2, T_3 respectively. The forces acting are $T_1 . BC$, $T_2 . CA$, $T_3 . AB$, perpendicular to these three edges respectively.

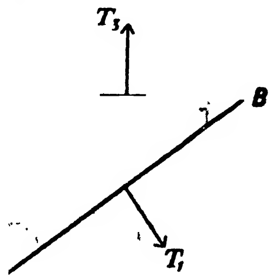


Fig. 131.

Resolve in the direction at right angles to AB, and we get

$$T_3 . AB = T_1 . BC . \cos ABC,$$
$$\therefore T_1 = T_3.$$

Hence the surface tension is the same in any two directions, *i.e.* is the same in all directions.

Consider next the equilibrium of a small rectangular portion $ABCD$. If we denote the tensions in AD, BC by T', T'' respectively, and resolve parallel to AB, we get $T' . AD = T'' . BC$, *i.e.* $T' = T''$.

Hence the surface tension is constant at all parts.

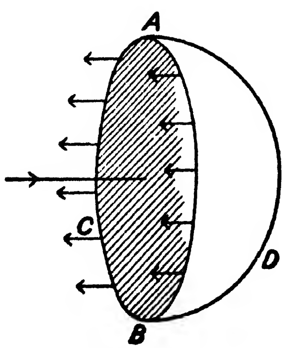


Fig. 132.

269. To find a Relation between the Radius of a Spherical Drop of a Liquid, the Surface Tension, and the Pressure.—Imagine the drop to be divided into two hemispheres and consider the equilibrium of one of them $ABCD$.

Let r = radius of the drop,
p = pressure at any point inside,
t = the surface tension.

The only forces acting on the hemisphere are (1) the thrust on the plane face ABC due to the pressure of the liquid

in the other half, (2) the tension of the surface acting round the edge of the circle ABC.

These forces are in equilibrium. Equating them we get $p \times$ area of $ABC = t \times$ perimeter of ABC.

$$p \cdot \pi r^2 = t \cdot 2\pi r$$
$$pr = 2t.$$

The pressure p is the pressure due to the surface tension only, and is the excess of the internal pressure over the external.

270. To find a relation between the radius of a Cylinder of Liquid, the Surface Tension and the Pressure.—Consider the equilibrium of the fluid contained between two planes at right angles to the axis. Call the circles in which the surface is cut ABC, $A'B'C'$. Divide the portion into two halves by a plane $BB'C'C$ passing through the axis of the cylinder. Consider the equilibrium of the half $ABCC'A'B'$.

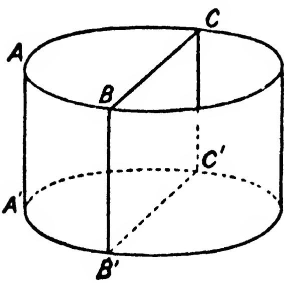


Fig. 133.

Let r = radius of the cylinder,
p = pressure inside,
t = surface tension,
h = distance between the planes of the two circles.

The external forces acting are (1) the thrust on the face $BB'C'C = p \cdot 2rh$, (2) the surface forces on the edges BB', CC'; each of these is equal to th, (3) forces on the edges ABC, $A'B'C'$ and thrusts on the faces ABC, $A'B'C'$. These thrusts are all parallel to the axis of the cylinder. Resolving in the direction perpendicular to the face $BB'C'C$ we get

$$p \times 2rh = 2th,$$
$$pr = t.$$

271. Curvature of Surfaces.—Articles 269, 270 are particular cases of a general theorem. Before this can be enunciated it will be necessary to draw attention to one or two geometrical results. Any two points P, Q on a

curve may be joined by a straight line (Fig 134). The position PT which the line PQ takes up when the points P, Q are indefinitely close to one another is that of a tangent to the curve.

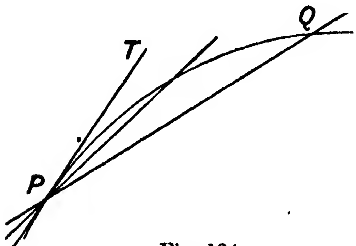


Fig. 134.

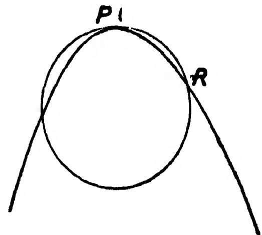


Fig. 135.

Now a circle can be made to pass through any three points. Let us suppose that P, Q, R (Fig. 135) are three points on a plane curve; draw a circle PQR through them. If P and Q approach one another indefinitely, the tangent to the curve at P and the tangent to the circle will be the same line. The circle will in this case touch the curve at P and cut it at R. Next let R

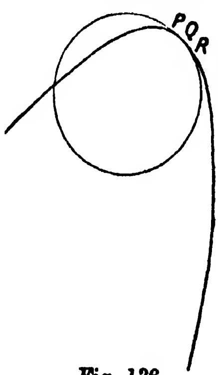


Fig. 136.

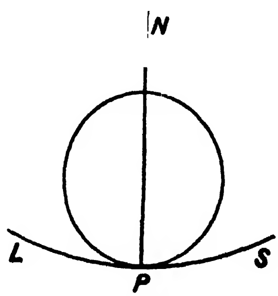


Fig. 137.

approach P (Fig. 136); then the circle PQR in its ultimate position when all three points are indefinitely close together is called the circle of curvature at P. It is a circle which, in the general case, not only touches the curve at P, but also crosses it at P.

Consider next the surface of a solid at any point P (Fig. 137), draw a normal at this point and let any plane $LPSN$ be drawn through the normal to cut the surface. The section will be a plane curve. The length of the radius of curvature at P of the section will depend on the direction in which the cutting plane $LPSN$ is taken. If the direction is such that this length is either a maximum or a minimum the section is called a principal section, and the radius of curvature of the section is called the principal radius of curvature of the surface. These two principal sections are always at right angles to one another. For the proof of this consult Smith's *Solid Geometry*.

The relation connecting the curvature of a surface, the pressure and the surface tension is

$$\frac{1}{r} + \frac{1}{r'} = \frac{p}{t},$$

where r, r' are the radii of curvature, t the surface tension, and p the difference of pressure between the two sides of the surface. This result may be obtained as follows:

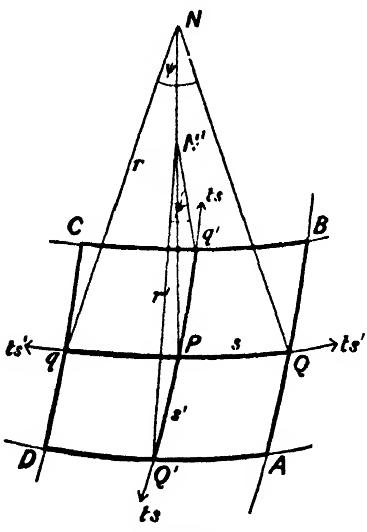


Fig. 138.

Let PN (Fig. 138) be normal to the surface and QPq be a principal section. When Q, q move up to P then the circle of curvature, centre N, will pass through qPQ.

Let $Q'Pq'$ be the other principal section, PN' its radius of curvature. Let a plane through Q perpendicular to PQ be drawn to cut the surface in the line AQB. Let planes be drawn in a similar way through q', Q', q: these will cut out an element $ABCD$ on the surface.

$$\text{Let} \qquad Qq = s, \qquad Q'q' = s'$$

$$\text{angle } qNQ = \psi, \quad q'N'Q' = \psi'$$

$$PN = r, \qquad PN' = r'.$$

Consider the equilibrium of the surface $ABCD$. The forces acting on it are (1) the thrust due to the liquid $= p \times ss'$, (2) the surface forces on the edges AB, CD, each of which is equal to $t.s'$, (3) the surface forces in the edges BC, AD; each of these is equal to $t.s$.

Resolve in the direction PN and we get

$$pss' = 2ts \sin\frac{\psi'}{2} + 2ts' \sin\frac{\psi}{2}.$$

But when q, P, Q are indefinitely close together

$$2s \sin\frac{\psi'}{2} = 2s.\frac{\psi'}{2} = s\psi' = s.\frac{s'}{r'}$$

and
$$2s' \sin\frac{\psi}{2} = \frac{ss'}{r}$$

$$\therefore\ pss' = \frac{tss'}{r} + \frac{tss'}{r'}$$

i.e.
$$\frac{1}{r} + \frac{1}{r'} = \frac{p}{t} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

272. Particular Cases.— The relation

$$\frac{1}{r} + \frac{1}{r'} = \frac{p}{t}$$

is general and holds for all surfaces. A few particular cases are of importance.

(1) If the surface under consideration is that of a sphere, then the two radii of curvature are equal and we get

$$\frac{2}{r} = \frac{p}{t} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

(cf. Art. 269).

(2) If the surface is that of a cylinder, one radius of curvature is equal to that of the cylinder and the other is infinite, one of the sections of a cylinder being a straight line. The equation then becomes

$$\frac{1}{r} = \frac{p}{t} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

(cf. Art. 270).

(3) If the surface is one of revolution and there is no difference of pressure the surface is a catenoid.

This is the case in which a soap film is supported between two parallel rings, the line joining the centres of

which is perpendicular to the planes of both. The relation governing the form is

$$\frac{1}{r} + \frac{1}{r'} = 0 \qquad\qquad (4).$$

Consider any plane curve LPR. Let it revolve about a line AB in its plane as axis and so trace out a surface of revolution. Let N be the centre of curvature for the curve at P. Let NP produced meet the axis AB in N'. Then N and N' are the two principal centres of curvature for the surface of revolution. If the surface is one that satisfies the relation (4) then $PN = N'P$. Now the only curve which satisfies this relation, *i.e.* is such that its radius of curvature at every point is equal to the intercept on the normal between the curve and a fixed straight line, is a catenary; the catenary being the curve in which a heavy flexible string hangs when supported at its ends. The surface of revolution is therefore a catenoid.

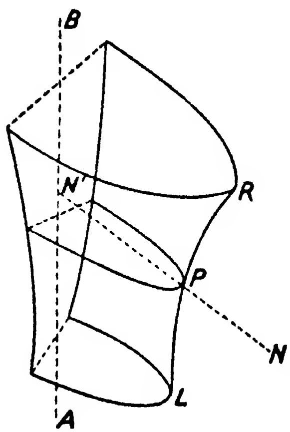


Fig. 139.

Hence if a soap film is symmetrical about an axis and has the same pressure on either side, its shape is that of the figure formed by the revolution of a catenary.

273. Possible Forms.—Surfaces other than catenoids are of course possible, but these are not symmetrical about an axis. These may be studied experimentally by dipping wires bent to different shapes in a soap solution. The resulting films thus obtained must always satisfy the relation $\frac{1}{r} + \frac{1}{r'} = 0$ at every point. In other words the total curvature must everywhere be zero. A plane film is the simplest example.

If the surface is one of revolution, then in all cases, whatever the pressure difference, the generating curve is the path traced by the focus of a conic rolling on a straight line. If the conic is a parabola the path traced by its focus is a catenary. This, as we have seen above, gives rise to a catenoid, a form possible only when there is no pressure difference between the two sides of the film.

The curve traced by the motion of the centre of a circle rolling on a straight line is another straight line. As the centre of a circle is also its focus, it follows that the straight line is a possible form for the generating curve. The surface of revolution is then a right cylinder.

274. Stability.—It is important to note that though many surfaces satisfy the relation $\frac{1}{r} + \frac{1}{r'} =$ constant, yet all are not really possible forms for the stable equilibrium of a film. The cylinder affords a good illustration of this point, for its equilibrium is only stable provided that its length is not greater than its circumference. To show this experimentally take two funnels equal in size, close the tube of one and connect that of the other to a rubber pipe; bring the edges together after dipping them in soap solution. When they are separated again there should be a film connecting the two rims. If the funnels are kept close together this film may be made to bulge outwards or inwards by blowing through the rubber tube, or the pressure may be so adjusted that the film is cylindrical. Separate the funnels and find the greatest possible length of the cylindrical film. This length can never exceed the circumference of the rim.

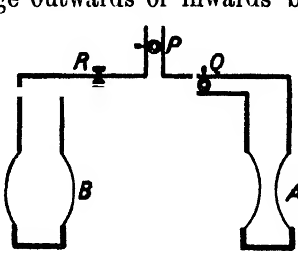


Fig. 140.

A pretty experiment due to Professor Rücker[1] explains the instability of the cylinder. The two films A and B (Fig. 140) are governed by the taps at PQR. If the tap Q is opened when the length

[1] See *Soap Bubbles* (by Professor C. V. Boys), published by S.P.C.K.

spanned by the film is less than the semi-circumference of the rings the air is squeezed out from B into A, so that both films become more cylindrical; when, however, the length spanned is greater than the semi-circumference, the air passes in the other direction, the neck in A gets narrower while B bulges out further. If we imagine that the film in A is connected to the end of B it is easy to see that A will expand while B contracts, so that they together form a cylinder, provided that the total length between the rings is less than the circumference. If the length is greater than this the figure departs further from the cylindrical shape and finally bursts.

275. Liquid Jets.—A jet of water falling from a circular orifice is at first cylindrical, but since this form is not stable it soon breaks up into separate drops. These drops have been illustrated and their forms explained in Art. 263. The length of the continuous portion of any liquid jet depends on the surface tension, the viscosity of the liquid, the velocity with which it issues, and the diameter of the orifice. If the velocity is great the length is also great, as in the case of the water in a fountain or from the nozzle in a fire engine. Viscosity also increases the possible length of the cylinder, and hence we can draw out very narrow threads of half-molten glass or quartz. The greater the surface tension the sooner does it cause the jet to break up. A jet of water escaping from a nozzle does not

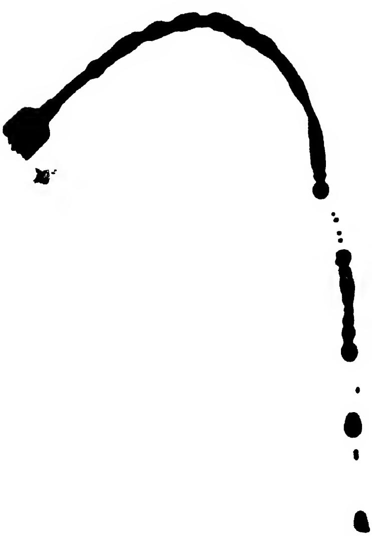


Fig. 141.

generally break up regularly and evenly. The drops are of different sizes, move with different velocities, and sometimes bounce up against one another (Fig. 141). This is due to slight accidental tremors to which the nozzle is subjected. These tremors impress necks on a cylindrical jet. They are irregular and at different distances from one another. Some of them disappear while others develop very rapidly, and break the jet up into drops. If, however, a regular disturbance is impressed on the nozzle the necks may be more evenly distributed, the drops will then form at the same place and be of the same size. A tuning fork placed on the stand which supports the jet produces this effect. To show it, a tube of glass may be drawn out and cut off square at the fine end. If this is connected to a reservoir some few feet above, a fountain can be obtained on a tray. The tuning fork makes the jet cease from scattering, and drops fall with a regular patter on the tray.

276. Three Separating Surfaces.—Two media in contact are separated by a single surface. With three media there are three different surfaces of separation, the tensions and energies of which have to be considered. Take the case of a drop of mercury resting on a horizontal plate of glass. The three media are glass, air, mercury; three surfaces, glass-air, air-mercury, mercury-glass. The shape of the drop depends on the intensity of gravity, the amount of mercury, and the values of the tensions of these three surfaces. The form ultimately adopted will be such as to make the potential energy a minimum. We shall prove first that the angle at which the mercury meets the glass is constant.

277. Angle of Contact.—Suppose PQ is a glass plate with a drop of mercury resting on it (Fig. 142). Let O be a point at the junction of the three surfaces. Since the glass is solid the point O must be somewhere in its fixed surface PQ, so that for equilibrium it is only necessary that the sum of the components parallel to PQ of the three surface forces be zero. If θ is the angle at which the

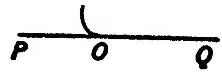


Fig. 142.

mercury meets the glass, and the surface tensions are called T_{ga}, T_{ma}, T_{mg}, then (Fig. 143)

$$T_{ma}\cos\theta + T_{ga} = T_{mg}.$$

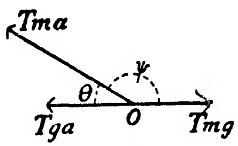


Fig. 143.

Hence θ is constant for the same three media.

The supplement ψ of this angle is usually called the angle of contact for mercury and glass.

278. No fluid media have been found which are such that three of them can be in contact along a line If three such media existed and O were a point in the line of intersection of the three surfaces we should have (Fig. 144)

$$\frac{T_A}{\sin\alpha} = \frac{T_B}{\sin\beta} = \frac{T_C}{\sin\gamma}, \text{ and } T_A,\ T_B,\ T_C$$

would be proportional to the sides of a triangle of angles π-α, π-β, π-γ: so that any two of the surface tensions must be together greater than the third. The condition is not satisfied by any known fluids. If oil, air, water satisfied, a drop of oil would float

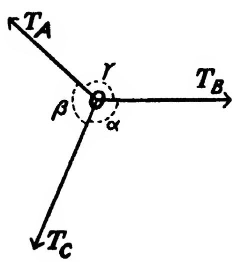


Fig. 144.

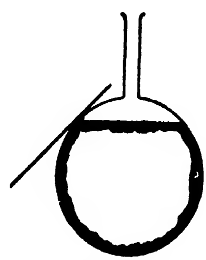


Fig. 145.

as a lenticular mass on the surface of water. As a matter of fact the oil spreads out in a thin film, the reason being that the water-air surface is stronger than the other two together.

279. Method of finding Angle of Contact.—The angle of contact may be found by taking a spherical flask and pouring in mercury (Fig. 145). When the depth is small the surface of the mercury is convex at the edges. This convexity decreases as the depth is increased until the surface becomes quite flat. If more mercury is added the

surface becomes concave. The obtuse angle made by the surface of the mercury when quite plane with the glass is the angle of contact. Another simple method is to take a sheet of glass partly immersed in a trough of mercury (Fig. 146). This must be tilted at such an angle that the under surface of the mercury is plane. The obtuse angle made by the glass with the horizontal surface of the mercury is the angle of contact.

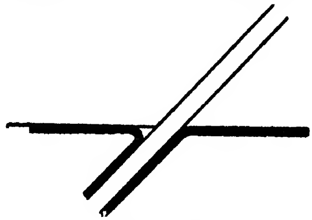


Fig. 146.

280. Rise of a Liquid in a Narrow Tube.—In a round very narrow tube (Fig. 147) the surface of the liquid is nearly spherical; call its radius R. Let the pressure at a point A just below the flat surface be called p_0: then that at B, just below the curved surface, is

$$p_0 - \frac{2t}{R}.$$

Now if h is the height to which the liquid rises the pressure difference must be $g\rho h$, if ρ is the density of the liquid.

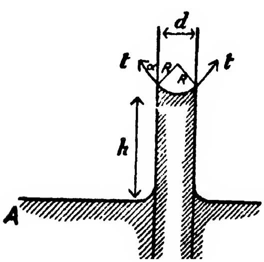


Fig. 147.

$$\therefore g\rho h = \frac{2t}{R}.$$

If a is the angle which the liquid makes with the tube

$$2R \cos a = d,$$

$$\therefore h = \frac{4t \cos a}{\quad}$$

Hence the rise of the liquid is inversely proportional to the diameter of the tube. In the case of a glass tube and water, the water wets the glass and we have

$$a = 0, \ \rho = 1,$$

so that $$h = \frac{4t}{gd}.$$

With mercury and glass the angle a is obtuse, so that $\cos a$ is negative. Hence h is negative, *i.e.* the mercury in the tube is depressed below the general level (Fig. 148). This explains why it is difficult to make mercury enter a fine thermometer tube.

281. Rise of a Liquid between two Parallel Plates.— Exactly the same method may be applied as in the preceding article, but in this case the curved surface is cylindrical: hence

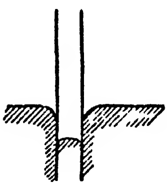


Fig. 148.

$$g\rho h = \frac{t}{R}, \text{ and } 2R\cos a = d,$$

where d is the distance between the plates.

This gives us

$$h = \frac{2t\cos a}{g\rho d},$$

so that the rise of the liquid is proportional to the distance between the plates.

282. The results of Arts. 280, 281 may also be deduced as follows :—

(*a*) *Rise in a capillary tube.*

Let a = angle at which the liquid meets the tube,

t = surface tension of fluid,

ρ = density of fluid,

h = height of liquid in the tube above the normal level,

d = diameter of the tube.

The weight of the liquid supported $= \pi\left(\frac{d}{2}\right)^2 hg\rho$. The junction of the liquid with the glass is a ring of circumference πd: the vertical component of the surface tension will therefore be $\pi d \,.\, t\cos a$.

$$\therefore \pi dt\cos a = \pi\frac{d^2}{4} g\rho h,$$

i.e.
$$h = \frac{4t\cos a}{g\rho d}.$$

(*b*) *Rise between parallel plates.*

Use the same notation, but let d denote distance between the plates.

Consider the equilibrium of a portion of the liquid between the plates, the horizontal length of which is one centimetre. The weight of this portion $= ghd\rho$. It is supported by the surface tensions: *i.e.* by two forces equal to t and inclined at an angle a to the vertical.

$$\therefore 2t \cos a = ghd\rho.$$

283. Rise of a Liquid between two Vertical Plates inclined at a Small Angle.—Consider two vertical glass plates which meet along the common vertical edge AB. The height at any point P will be given by the relation of Art. 281 so that $hd =$ constant. But the distance between the plates is proportional to the distance, l say, of P, from the edge AB. Therefore $hl =$ constant. P must therefore lie on a rectangular hyperbola of which AB, AX are axes.

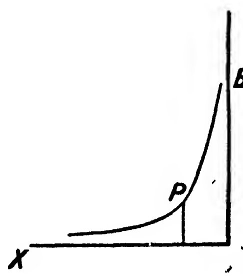


Fig. 149.

284. Attraction or Repulsion between Parallel Plates.—In the case of the plates in Art. 281 the pressure in the liquid between the plates just below the concave surface is less than that of the surrounding atmosphere. This may be deduced as follows:

Pressure at $B =$ pressure at C on the same level
$=$ atmospheric pressure.

Also pressure at $B =$ pressure at $A +$ pressure due to depth AB of liquid, therefore pressure at A is less than atmospheric.

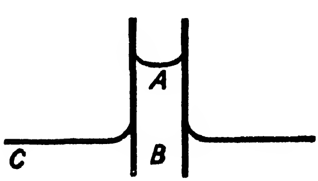


Fig. 150.

It follows from this that the pressure of the atmosphere tends to thrust the two plates together.

(If there were no external pressure due to the atmosphere the plates would still be pulled towards one another, for the liquid between the plates would exert a negative pressure, *i.e.* a tension. Cf. Art. 226.)

If the liquid is one which does not wet the plates the result is still the same, for the level of the liquid is depressed and the sign of the curvature of the surface is changed. On the other hand if the plates are of different materials such that one only is wetted by the liquid the two plates are pushed away from one another. The reason for this is that the height to which the liquid rises on the inside of the wetted plate is less than that to which it rises on the outer side, and the depth to which it is depressed by the other plate on the inside is less than on the outside (Fig. 151). A consideration of the forces acting will show that the tendency is for the plates to separate. These results account for the way in which bubbles in a cup of tea gather round the edges, or group themselves up in little bunches. In the same way straw and sticks float together on the surface of a pond. It is well known that the skin of water is sufficiently strong to support a dry needle laid gently on the surface. The water in this case does not wet the needle. We may therefore expect a needle floated near the edge of a bowl of water to find its way gradually towards the centre. This actually occurs. A wetted chip of wood behaves in exactly the opposite way. If the two are floated near one another they will gradually separate: the needle finding its way towards the centre, and the chip moving to the edge.

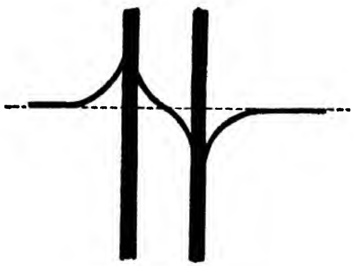


Fig. 151.

285. To find the Surface Tension of a Liquid.—A capillary tube may be used to find the surface tension of a liquid. To do this it is necessary to use a clean tube of fine and uniform bore. This is held vertical and dipped into the liquid. The height to which the liquid rises is measured, and the diameter of the tube is carefully measured by a microscope or by a thread of mercury. The

result is then obtained from equation of Art. 280. The tube may be used to compare the tensions of two liquids, *e.g.* water and alcohol. In this case it is not necessary to know the bore of the tube.

The maximum depth of a pool of mercury resting on glass

$$= 2\cos\frac{a}{2}\sqrt{\frac{t}{g\rho}}.$$

This relation may be used to find an approximate value of t. Other methods are indicated later.

286. Evaporation at Curved Surfaces. Formation of Clouds.—Consider the case of a vertical capillary tube placed in a liquid. In order that we may neglect the effect of the atmosphere let us imagine the whole to be placed in an exhausted receiver. When equilibrium is established this receiver will be filled with vapour.

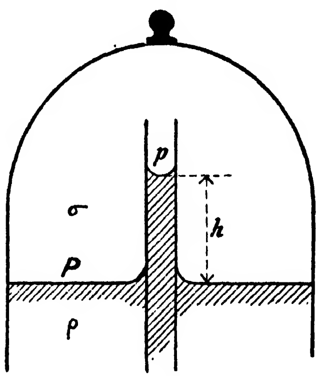


Fig. 152.

Let h = height to which the liquid rises in the tube. Let σ, ρ be the densities of vapour and liquid respectively. Let P = the vapour pressure just above the horizontal surface of the liquid.

Then the pressure due to the vapour just above the concave surface in the tube will be $P - g\sigma h = p$ (say).

The pressure in the liquid just below the meniscus is $P - g\rho h$.

If we can regard the meniscus as a portion of a sphere of radius r, then (Art. 269) the difference in pressure between the two sides of the surface is equal to $\frac{2t}{r}$, where t is the surface tension.

$$\therefore p - (P - g\rho h) = \frac{2t}{r},$$

$$p - P + \rho\frac{P-p}{\sigma} = \frac{2t}{r}, \qquad i.e. \qquad P - p = \frac{\sigma}{\rho - \sigma}\cdot\frac{2t}{r}.$$

Now the vapour both above the curved surface and above the plane surface is saturated: it follows then that the saturation pressure for a vapour in contact with a concave liquid surface is less than that of a vapour in contact with a plane surface, and, further, that vapour will condense more readily on a concave surface than on a plane surface. This result might be verified by closing the bottom of the tube and removing some of the liquid from it. Condensation would then begin on the liquid inside the tube and continue till the liquid reached its former level.

Carrying the argument one step further leads to the conclusion that condensation on the surface of a liquid is more rapid the less the convexity of the surface. Hence in a cloud the tendency is for the vapour from little drops of water to condense on larger drops, so that big drops form at the expense of little ones.

Further, take the case of a very tiny drop of water in a space filled with moisture at the dew point. The moisture will not condense on this because the pressure necessary for condensation on a very convex surface is great. On the contrary the drop would evaporate. Hence it is possible to have vapour cooled below the saturation point, for the fine drops cannot form. If, however, dust is present, this offers a flatter surface, and condensation is more easily brought about. In towns particles of smoke form suitable nuclei for the deposition of moisture, and fogs are the result. If there were no smoke, the moisture would still condense, as dust in sufficient quantity would always be present.

Electrons serve as mist nuclei. Professor Sir J. J. Thomson made use of this property to count the number of them in a given charge.

The effect of dust may be shown experimentally as follows:— Smear a little glycerine on the plate of an air pump and place there a small vessel containing a little water. Cover up with a glass bell jar. This must be left for a considerable time, say a day, for the dust in the jar to settle on the glycerine. The air in the bell jar will be saturated with moisture, so that a slight lowering of temperature might be expected to produce condensation. To bring about the fall in temperature give a few strokes with the pump. The air inside the jar will expand and cool, but no fog will be seen. Now readmit air and introduce a little dust into the receiver by burning a match under the glass. A stroke or two of the pump now produces the fog

287. Ripples.—If two fluids in contact move with different velocities the surface of separation becomes puckered up into undulations. These undulations are governed partly by gravity and partly by surface tension. Waves and ripples are also formed when the surface of water is disturbed in any way, as for instance by a boat or by the fall of a stone. The velocity with which a wave travels in a liquid of great depth depends on the wave length, and it may be shown

that
$$v^2 = \frac{\lambda g}{2\pi} + \frac{2\pi T}{\lambda\rho}$$

when v = velocity of propagation,
λ = wave length,
T = surface tension,
ρ = density of the liquid.

The equation is a quadratic in λ. Solving it we get

$$\lambda = \frac{v^2 \pm \left\{ v^4 - \dfrac{4gT}{\rho} \right\}^{\frac{1}{2}}}{g/\pi}.$$

From this we see that v^4 cannot be less than $\dfrac{4Tg}{\rho}$.

Hence the least possible value of v is given by the relation

$$v = \left\{ \frac{4Tg}{\rho} \right\}^{\frac{1}{4}}.$$

In the case of waves on the surface of water, the value of T is about 74 dynes per cm. This gives 23 cm. per second as the minimum velocity corresponding to a wave length of about 1·7 cm. Waves of length less than this critical value are termed ripples.

Ripples are therefore mainly dependent on surface tension, waves on gravity. The longer a wave is the greater its velocity, the less the wave length of a ripple the greater its velocity.

The above equation gives us a means of finding T, provided that λ, v are known. By means of a vibrating tuning-fork a regular train of ripples can be generated,

and the number of these in a given length can be ascertained. This gives the value of λ. Also $v = \lambda n$, where n is the frequency of the fork. Hence T can be found.

288. Temperature.—The value of the surface tension of any liquid is dependent on the temperature, and decreases as the temperature rises. At the critical point the surface tension vanishes, so that there is no definite line of separation between gas and liquid.

289. Electrical Condition.—Electrical polarisation of a surface alters its tension. To show this pour a little mercury into a beaker containing a solution of salt or acid. Let a glass tube, having one end drawn out to a very fine point, be held vertical with the fine end dipping into the solution, and let mercury be poured into the other end. It will be prevented from running out by the surface tension. Now join these portions of mercury to places of different potential, the first being, say, half a volt higher than the second. The effect of this will be to make the mercury in the capillary tube rise, showing that its surface tension has increased.

290. Laplace's Theory.—The exact nature of the forces which give rise to the phenomena of surface tension is not fully understood, but it is possible that the forces acting between neighbouring particles (or molecules) near the surface of a liquid are not essentially different from the forces between particles in the interior. Laplace's theory of capillary attraction assumes that forces exist between molecules which are near one another, but that these forces are insensible between particles separated by a sensible distance. A particle of water at or very near to the surface of separation from air would in this case be acted on mainly by forces from the lower side, there being very few or no particles above it. Since the forces are supposed sensible only at insensible distance, this want of symmetry can only exist extremely close to the surface. At points in the interior of the water the forces are symmetrical and so less easily detected. Their existence, however, must be postulated to account for the phenomena of cohesion in liquids (*v.* Art. 226).

Examples XV.

1. Find pressure in the interior of a raindrop of which the radius is one millimetre.

2. Show that the vapour pressure of a liquid in the form of a drop of radius r exceeds the vapour pressure over a flat surface of the same liquid by $\frac{\sigma}{\rho} \cdot \frac{2T}{r}$, where σ is the density of the vapour, ρ that of the liquid, and T is the surface tension.

What do you suppose to be the function of dust nuclei in the formation of a cloud?

3. Give some account of the determination of surface tension of liquids by the method of ripples.

4. A capillary glass tube with conical bore is dipped with its apex upwards into a liquid. Explain how you would determine the height to which the liquid rises in the tube.

5. Explain what is meant by the surface tension of a liquid, and describe some method of finding its value for a soap film.

6. Two equal bubbles coalesce. Show how to determine the radius of the resulting bubble in terms of those of the original bubbles, and of the surface tension of the film.

7. Investigate why a long cylinder of liquid is unstable from the effects of surface tension, and give some instances of the effects of this instability.

8. A flexible membrane is exposed to hydrostatic pressure. What is the relation between the pressures on its two sides, its principal tensions, and its principal curvatures?

9. A small quantity of a certain fluid which is lighter than water is placed on the surface of water in the presence of air. What condition determines whether the liquid will spread out into a thin film or form a lenticular drop, and in the latter case what condition determines the angle of the lens?

10. Give any explanation which you can of surface tension in accordance with the molecular theory of the constitution of bodies and the principle of the conservation of energy.

11. How does the normal pressure due to a stretched film depend on the curvature of the film and its tension?

12. A soap bubble is formed between two equal wire rings and the rings are then separated, their axes being kept coincident so as to draw the bubble out to form a cylinder, the ends of which are closed by spherical films. What is the ratio of the radii of the two portions?

13. Prove the relation of Art. 268 by taking a triangle of any shape.

14. A sphere of water of radius 1 mm. is sprayed into a million drops all of the same size. Find the work expended in doing this.

15. Give some account of the determination of the surface tension of liquids by the method of ripples.

16. Define the surface tension of a liquid and show that it is equal to the energy per unit area of the surface. Calculate the work done on the film in blowing a soap bubble from a diameter of 3 cm. to one of 30 cm. if its surface tension be 45 in C.G.S. units.

17. Explain the conditions that determine the shape of a quiescent drop pendant from the end of a vertical tube of circular cross section.

18. The surface tension of water is 72 dynes per cm. Calculate how far water will rise up a circular tube ·2 cm. in diameter.

19. A soap bubble is spherical in shape, and has a diameter of 10 cm.; if the surface tension of the surface separating soap solution and air is 20 C.G.S., what is the pressure of the air in the bubble?

20. A cylindrical boiler is built with spherical ends. If the diameter of the cylinder is 5 feet, find the tensions in the different parts of the boiler when the pressure of the steam is 200 lb. to the square inch.

21. A metal bar can stand a tension of 12,000 per square inch. Find what pressure a hollow sphere, radius 1 ft., thickness $\frac{1}{5}$ inch, can support without bursting.

22. Prove the theorem of Art. 278 by considering the energy spent in displacing the line in which three fluids meet. Suppose the plane of the paper meet this line at right angles at O, and the surfaces in lines OA, OB, OC.

Denote the surface energy of the three surfaces by S_a, S_b, S_c.

Let a SMALL displacement be made in which the point O moves to O', the angle AOO' being right: the new sections of the surfaces are called AO', BO', CO'.

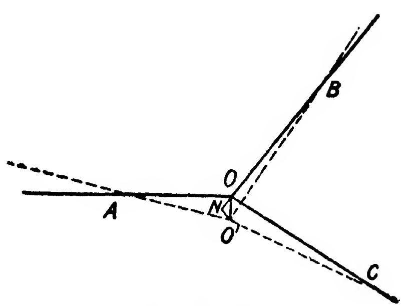


Fig. 153.

The increase in the first surface is zero, for $AO = AO'$ if we neglect small quantities of the second order.

Draw $O'N$ perpendicular to BO.

The increase in $BO = BO' - BO = ON = OO' \sin AOB$.

The increase in $CO = -OO' \sin COA$.

Hence $$S_2 \sin AOB - S_3 \sin COA = 0.$$

Since the total increase in energy due to any small displacement must be zero,

$$\therefore \quad \frac{S_2}{\sin COA} = \frac{S_3}{\sin AOB}$$

$$= \frac{S_1}{\sin BOC} \text{ by symmetry,}$$

i.e. $$\frac{T_1}{\sin \alpha} = \frac{T_2}{\sin \beta} = \frac{T_3}{\sin \gamma}.$$

MISCELLANEOUS EXAMPLES.

CHAPTER I.

1. A force acting on a mass of 10 pounds produces in it a velocity of 5 feet per second in one minute. Express the force in dynes.

2. Find the number of watts in one horse-power, given
1 ft. $= 30{\cdot}48$ cm., 1 lb. $= 453{\cdot}6$ gm., $g = 981$ cm./sec.^{2}.

3. If the fundamental units are the velocity of light in air, the acceleration of gravity at Greenwich and the density of mercury at 0° C., find the units of length, mass, and time. (Take velocity of light $= 3 \times 10^{10} \dfrac{\text{cm.}}{\text{sec.}}$, acceleration of gravity $= 9{\cdot}81 \times 10^2 \dfrac{\text{cm.}}{\text{sec.}^2}$, density of mercury $= 13{\cdot}6 \dfrac{\text{gm.}}{\text{cm.}^3}$.)

4. Show that when bodies of geometrically similar form and of the same material differing only in dimensions vibrate in the same manner, the vibrations being due to the elasticity of the material, their periods are proportional to their linear dimensions.

CHAPTER II.

1. A spherometer reads ·460 mm. when its legs are on a plane surface and 2·735 mm. when on a spherical surface. The distance from leg to leg is 43·56 mm. Find the radius of the surface.

2. A circular scale reads to 10 minutes of arc. It is required to read to 10 seconds of arc. How must the vernier be constructed?

3. Show how to find the area of an irregular plane figure, and taking a semicircle of 5 cm. radius, find the area between the curve and the diameter by the usual formula and by Simpson's two rules. (Draw ordinates at every cm. along the diameter.)

CHAPTER III.

1. Masses of 10 gm. and 1 gm. are connected by a stretched elastic string and placed on a smooth horizontal table. If they are released simultaneously, compare their velocities at any instant before collision occurs.

2. A column of water 10 metres long is moving along a pipe behind and in contact with a piston of 15 cm. diameter moving with a velocity of 450 cm. per sec. A block in the pipe causes the piston to stop dead in 0·1 sec. Find the time average of the force due to the stoppage.

3. A ball of mass 4 lbs. is moving northwards on a smooth level table with a velocity of 6 ft. per sec. It collides with a stationary ball and after the collision moves eastwards with a velocity of 5 ft. per sec. Find the impulse given, and assuming that the collision lasted 0·002 sec., find the average force of the blow.

Chapter IV.

1. How would you proceed to find the density of dry air at 0° C. and 760 mm. pressure?

2. If a body weighs m_1 gm. in air of density d and m_2 gm. in a liquid of density D, show that its density

$$= \left(\frac{m_1 D}{m_1 - m_2} - \frac{m_2 d}{m_1 - m_2}\right) \text{ gm. per c.c.}$$

3. The specific gravity of sea water is 1·028 and of ice ·918. Find what fraction of an iceberg is above water.

4. An exhausted glass tube was weighed at a time when the barometer stood at $72\frac{1}{2}$ cm.; and 100 gm. exactly balanced it. On the next day when the barometer had risen to 78 cm. the globe was filled with hydrogen at atmospheric pressure, and 100 gm. again balanced it. Neglecting the volume of the glass and weights, find the specific gravity of hydrogen referred to air.

5. A glass flask weighing 20 gm. and holding 100 c.c. is to be sunk in water with mouth downwards and open by means of a lead weight tied by a string to the neck. What mass of lead will be necessary? If at a depth of 100 metres the lead becomes untied, will the flask come up again? (Sp. gr. of glass = 2·5, sp. gr. of lead = 11·0, atmos. press. = 10^6 dynes/cm.^{2}.)

Chapter V.

1. Trace as far back as you can through its various transformations the energy obtained from the water in a water-wheel.

2. A bullet weighing 25 gm., and moving with a velocity of 300 metres per sec., is stopped by impact against a bone, being brought to rest within a distance of 3 cm. from first striking. Calculate the average force exerted by the bullet on the bone.

3. Show that if a piston moves along a cylinder against a constant pressure the work done in a stroke is equal to the pressure multiplied by the volume swept out by the piston.

4. What horse-power is required to raise enough water to cover 100 acres of land a day to the depth of one inch, if the land is 60 feet

above the river? If a steam-engine is employed whose efficiency is $\frac{1}{30}$, and the energy of 1 lb. of coal is 12×10^6 foot pounds, how much coal will be consumed per day?

5. When a wheel is being turned around on an axle, show that work done in turning it can be determined by measuring the moment M of the turning effort required to prevent the axle from rotating and then calculating the work W done in n revolutions by the formula $W = 2\pi nM$. How would you measure M experimentally? In what units must M be expressed if W is to be expressed in foot-pounds?

6. Electrical energy is sold at 4d. per kilowatt-hour. The mechanical equivalent of the heat given out by the burning of coal worth 4d. is 10^8 foot-pds. Compare the prices of the two forms of energy. Why is electrical energy so much dearer than coal energy?

Chapter VI.

1. A railway curve of 800 yards radius is traversed by a train going at a speed of 20 miles an hour. If the distance between the rails is 5 feet, how much should the outer rail be raised in order to avoid the risk of the train leaving the rails?

2. A certain string will break under a load of 50 kilogm. A mass of 1 kilogm. is attached to the end of a piece of this string 10 metres long and is rotated in a horizontal plane. Find the greatest number of revolutions per minute which the weight can make without breaking the string.

3. A stone of mass 10 lbs. is revolving in a vertical circle at the end of a string 8 ft. long., the other end of which is fixed. When the stone is at the top of its circle, its velocity is 16 ft. per sec. Assuming $g = 32$, find the stretching force in the string when the stone is at the top and bottom of its circle, and also when it is at a level with the centre.

4. A rope whose mass is $\frac{1}{4}$ lb. per foot is made into a ring 3 ft. in diameter and caused to rotate in a horizontal plane about the centre of the circle. If the greatest load the string can bear is 800 lbs. wt., find the maximum number of revolutions the string can make per minute.

5. A sphere of mass m rolls with a velocity v along a plane; find the total kinetic energy. If a sphere rolls (starting from rest) in 5·3 seconds along a plane 1 metre in length, of which the upper end is raised 1 cm. above the lower, find the acceleration of gravity.

Chapter VII.

1. A heavy ball of mass M and radius r is suspended by a light string of mass m and length l. Find the time of swing of the pendulum thus formed.

2. A circular hoop is hung on a nail and left swinging in its own plane. Show that it would keep time with a simple pendulum of length approximately equal to its diameter.

3. Show how the moment of inertia of a bar may be found by observation of its period of vibration when supported by a bifilar suspension.

4. A rod of mass m and of length a is suspended by two vertical strings of length l attached to its ends. It is given a slight displacement from its position of rest and released. Find its period of oscillation.

5. Show how to find the value of g by the method of oscillation of a suspended spiral spring loaded with a mass which when at rest pulls the spring down a distance a.

6. A ball of known radius r rolls to and fro without slipping inside a spherical surface. From the periodic time T show how to calculate the radius of the surface.

7. Explain why the cushion of a billiard table is made to receive the impact of the ball at a height $\frac{7}{10}$ths of the diameter of the billiard ball.

8. If a body execute a simple harmonic vibration in time T_1 under one constraining force, and in time T_2 under another, what will be its time under both forces together?

Chapter VIII.

1. Define the dynamical mean sun and the mean sun, stating at what points they have the same celestial latitude and when the former coincides with the true sun. Show that the mean sun has a uniform diurnal motion and state how it measures mean time.

2. Define the equation of time. Of what two parts is it generally taken to consist? State when each of these parts vanishes, is positive, or negative. Give roughly their maximum values, and sketch curves showing their variations graphically.

3. If on a certain day the sun-dial be 10 minutes before the clock, what is the value of the equation of time on that day? Will the forenoon of that day or the afternoon be longer and by how much?

Chapter IX.

1. Explain the terms Young's modulus, rigidity, compressibility. Explain how the coefficient of rigidity may be determined experimentally.

2. Find the greatest length of steel wire that can hang vertically without breaking. (The breaking stress of steel = $7 \cdot 9 \times 10^9$ dynes/cm.^{2}; the density of steel is $7 \cdot 9$ gm./c.c.)

3. Explain what is meant by Poisson's ratio. How can it be found experimentally for a body like iron? If its value for a body is found to be $\frac{1}{2}$, what does this show?

4. A uniform glass tube is hung from a support and stretched by a weight. It is found that 1 metre of tube stretches by ·06 cm., but that a column of water 1 metre long contained within the tube lengthens by only ·04 cm. Find Poisson's ratio for the glass.

5. If the equation $\frac{1}{Y} = \frac{1}{9k} + \frac{1}{3n}$ is used to find k for a metal of which Y is about 2×10^{12} and n is about 10^{12}, these quantities being determined with errors of the order of 1 per cent., what may be the error in k?

6. How can the variation in length of short thick steel bars with tension and compression be found experimentally?

7. Explain how the value of the Young's modulus of a substance may be found by observations on the depressions of the middle point of a rod of this substance when supported at the ends and loaded in the middle.

8. A brass bar 1 metre long and 5 mm. square in section is supported horizontally at its ends. A load of 100 gms. is hung on the middle of the bar and the depression is observed to be ·41 cm. Find Young's modulus for brass.

9. Describe one method of finding experimentally the modulus of rigidity of a solid, and give the theory of the method.

Make a rough estimate of the maximum probable percentage error in the modulus deduced from a single set of observations with the apparatus you describe.

10. Calculate the time of vertical oscillation of a mass of 1 kilogm. hanging by a steel wire 3 metres long and ·5 mm. in diameter. (Y for steel $= 2 \times 10^{12}$ C.G.S. units.)

11. The balance wheel of a watch may be considered to be a heavy rim of 1 cm. radius. Its mass is 1·2 gm. Find the couple of restitution per unit angular displacement which the spring must exert in order that the period of complete rotation must be 1 sec.

12. A steel spring 300 cms. long is wound on a cylinder 2·8 cm. in diameter, the diameter of the wire being ·2 cm. It is loaded with a kilogramme. Find the depression produced. (Take $n = 8 \times 10''$ dynes per sq. cm.)

13. Explain what is meant by the conservation of momentum.

Two equal masses are attached to the ends of a string passing over a light frictionless pulley. One of the weights is supported on a small table while the other is raised 10 cm. and let fall freely through that distance. Find the velocity of the two weights after the string becomes tight.

14. An ivory ball dropped from a height of 100 cm. upon a cast-iron surface rebounds to a height of 50 cm. Find the coefficient of restitution between these two materials.

Chapter X.

1. Give an account of some method by which the mean density of the earth has been found accurately. Give an account of one method of finding the constant of gravitation G. Given $G = 6 \cdot 7 \times 10^{-8}$, the radius of the earth $= 6 \cdot 4 \times 10^{8}$ cm., and its mean density $= 5 \cdot 5$ gm. per c.c.: calculate the acceleration of gravity at the earth's surface.

2. Explain how to find the value of g by observations on a pendulum which swings in nearly, but not exactly, equal times from two points of suspension. Deduce your formulae from first principles.

3. How has the pendulum been used to find the form of the earth?

4. Gravity at the poles exceeds gravity at the equator in the ratio 301 : 300. A pendulum regulated for the poles is taken to the equator. Calculate how many seconds a day it will gain or lose.

5. How many times faster than the present speed would the earth have to rotate on its axis in order that the apparent weight of bodies at the equator should be zero?

6. Describe the methods adopted and the precautions necessary in making an accurate gravitational survey of a country with pendulums or other apparatus.

A simple pendulum whose bob has a specific gravity of 8·5 has a period 0·5050 sec. in vacuo. When swung in air it behaves as though each element of volume had its mass increased by twice the mass of the air displaced. Find the change of period when the pendulum is swung in air of specific gravity 0·0012.

Chapter XI.

1. If a medium of density ρ at a pressure p is moving with velocity $\pm v$, show that the momentum transferred per sec. per sq. cm. across a surface perpendicular to v is $p + \rho v^2$.

A circular cylinder of length 100 cm. and diameter 10 cm. is moving parallel to its axis under the action of a force of 10^6 dynes applied at one end. Calculate the pressure at a point 75 cm. from the end at which the force is applied.

2. Neglecting effects due to the motion of the molecules show that the mean free path of a molecule of a gas is equal to $\dfrac{1}{4\pi n r^2}$ where n is the number of molecules per c.c. and r the radius of each molecule. Show also that the mean free path of a particle of negligible dimensions is $\dfrac{1}{\pi n r^2}$.

3. Show from the kinetic theory of gases that the number of molecules per unit volume is the same for all gases at the same temperature and pressure.

4. How has the viscosity of air and other gases been determined? How is viscosity explained on the kinetic theory of gases? Show how the mean free molecular path may be obtained when the viscosity is known, and find the mean free path in air, taken as a uniform gas; given that the density of air $= 1\cdot2 \times 10^{-3}$ gm./cm.^{3} at 0°C. and a pressure of 10^6 dynes/cm.^{2}, and the coefficient of viscosity $= 1\cdot7 \times 10^{-4}$ dyne/cm.^{2} per unit velocity gradient.

5. What is meant by the characteristic equation of a perfect gas? Explain, giving a sketch, the variations in the behaviour of, say, carbonic acid (gas) from the ideal so expressed.

6. Describe any experiments which have been made to determine the variation of gases from Boyle's Law.

7. One hundred cubic centimetres of air at a pressure of 10^6 dynes/cm.^{2} is compressed isothermally to half its volume. Assuming that air obeys Boyle's Law find the work done.

8. A closed porous pot filled with air is provided with a manometer. Describe the indications of the manometer if the pot is suddenly surrounded by and kept in (*a*) coal gas, (*b*) carbon dioxide. Give some explanation on the kinetic theory.

9. A bottle containing 1 litre of hydrogen at atmospheric pressure is closed by a plate of graphite cemented over the mouth, and is placed in a very large vessel filled with oxygen also at atmospheric pressure. What is the pressure, sp. gr. relative to hydrogen, and percentage composition by weight of the gas in the bottle when $\frac{1}{100}$ of the hydrogen has diffused out?

10. A bicycle tyre leaks at the valve, where it is found that if wetted a bubble one quarter of an inch in diameter is blown in about 60 seconds when the excess pressure is two atmospheres. How long will it take the excess pressure to drop to one atmosphere? The inner diameter of the air tube may be taken as $1\frac{1}{4}$ inch and the diameter of the axis of the air tube as $26\frac{1}{2}$ inches.

11. The mercury in a barometer containing some air stood at a height of 70 cm., and the volume of the tube above the mercury was 20 c.c. The tube was then lowered into the mercury reservoir until the volume above the mercury was 10 c.c., when the barometer indicated 65 cm. only. Calculate (1) the true barometric height and (2) what the reading of the barometer in question would be if its tube were raised until the volume above the mercury became 100 c.c.

12. A vertical tube is placed with its lower end under mercury. The upper end is closed. When the barometer stands at 30 inches the mercury is at the same level inside and out, and the tube which contains air rises 60 inches above the mercury. What is the height of the barometer (to the nearest tenth of an inch) if, when at the same temperature, the mercury has risen one inch inside the tube, the level outside being maintained the same as before?

13. Give an account of any method available for the measurement of small gaseous pressures below 0·1 mm. of mercury. What limits the accuracy of the method you describe?

Chapter XII.

1. A man 1·7 metre high changes from the vertical to the horizontal position. If the density of the blood is 1·03 gm. per c.c. calculate the change in blood-pressure at his head, assuming that it stays constant at his feet.

2. A tank full of water is bounded by rectangular sides. Find the depth of the centre of pressure of any one side.

3. A U-shaped tube has one arm open to the earth and the other connected to a gas receiver. Show that the difference of pressure in the receiver and the atmosphere can be read by the difference of levels in the two arms. Why should the tube have uniform radius?

4. A fixed vertical glass U-tube of uniform section contains a liquid which can move in it without friction. If the levels of the liquid be disturbed show that the period of the swings is the same as that of a simple pendulum whose length is half the total length of liquid in the tube, and that it is independent of the density of the liquid.

Chapter XIII.

1. How has the bulk modulus for liquids been determined experimentally, and what do you know of the results obtained?

2. Describe the way in which the different parts of a viscous liquid move when flowing slowing through a fine tube. What change in the motion takes place if the motion is rapid? How would you propose to study these motions experimentally?

3. In an experiment with Poiseuille's apparatus the following figures were obtained:—Volume of water issuing per minute = 7·08 c.c. Head of water = 34·1 cm. Length of tube 56·45 cm. Radius of tube = ·0514 cm. Find the coefficient of viscosity.

4. The space between two parallel discs is filled with a liquid of viscosity μ. Find the couple on a circular area at the centre of one of the discs when the other is kept rotating with uniform angular velocity ω.

5. Define the (dynamical) viscosity of a substance and find its dimensions.

There are two concentric (coaxal) cylinders, A and B, each 10 cm. long. The inner one, A, is 10 cm. in diameter and there is a space of ·5 mm. between the outside of A and the inside of B. B is fixed and a continuous couple of 2490 dyne-cms. applied to A keeps A rotating 15 times per second. Calculate the viscosity of air.

6. Describe some experiments to illustrate the diffusion of liquids. How can diffusion be explained on the molecular theory, and how would you expect the rate of diffusion to vary with the temperature?

7. If a lump of sugar be held just below the surface of tea in a cup, it dissolves much more rapidly than it does if it is allowed to drop to the bottom of the cup, but not so fast as it does if it is well stirred. How do you account for this?

8. Define the coefficient of diffusion of a salt in a solution and explain how it can be found experimentally.

9. Explain how the osmotic pressure of a substance in solution may be determined, and give a short description of the theory of the phenomenon.

10. Find the depression of the freezing point of a water solution of strength 1 gm.-molecule per litre of a substance which does not dissociate. (Assume atmospheric pressure = $1{\cdot}014 \times 10^6$ dynes per sq. cm., $L = 80$ calories per gm. of water, and $J = 4{\cdot}2 \times 10^7$ ergs per gm. calorie.)

Chapter XIV.

1. A train, of which the mass is 200 tons, can be drawn by an engine at the uniform speed of 30 miles an hour up an incline of 1 in 200, or at 40 miles an hour up an incline of 1 in 400. Assuming the frictional resistances to be independent of the velocity, calculate the frictional resistance in lbs. per ton, and the horse-power exerted by the engine, which is to be taken as the same in both cases.

2. Two unequal masses, M and m gm. respectively, are connected by a belt which passes around a pulley one and a half times, the masses hanging down one each side of the pulley. The pulley is made to rotate at such a speed that the masses hang motionless. Find the work done when the pulley makes n revolutions, the pulley being of radius r cms. If $M = 4000$ gm., $m = 232$ gm., circumference of pulley $= 47{\cdot}65$ cm., $n = 552$, find the work done. Where does the energy go?

3. A billiard ball struck on a level with the centre has an impulse which makes it slide; friction with the table causes it to roll. How far will it move before the motion becomes pure rolling without any slip? Assume its mass $= m$, its initial velocity $= u$, and force of friction $= F$.

4. A cord is wrapped one and a half times around a horizontal rod of circular cross section and a mass of 60 gm. is hung from one end. What mass is required at the other end of the cord to keep the 60 gm. from slipping down? (Take $\mu = {\cdot}3$.) What is the least load required to pull the 60 gm. up?

Chapter XV.

1. If a liquid trickle gently over a polished metal sphere, and collecting at the lower surface fall off drop by drop, what are the conditions which determine the size of the drops?

2. Explain why oil forms a film between water and air, and from this fact show that water will gather into drops upon a greasy surface.

3. The pressure of the air in a soap bubble of diameter 7·0 mm. is 8 mm. of water above atmospheric. Calculate the surface tension of the soap solution.

4. A capillary glass tube with slightly conical bore is dipped with its apex upwards into a liquid which wets it. If a *very* small hole is left at the apex so that the air can escape, find an expression in terms of the length l of the tube, its radius r at the lower end, the surface tension T, the density d of the liquid, and the local value of g, for the height h to which the liquid will rise, and for the smallest value of the surface tension which will make it rise to the top.

5. A conical glass tube, 20 cm. long, 0·3 cm. in diameter at one end, and 0·1 cm. in diameter at the other end, is fixed vertically with its larger end just touching the surface of some water whose surface tension is 80 dynes per cm. Calculate the height to which the water will rise in the tube.

6. Two vessels are in communication by a small conical tube whose axis is horizontal and the diameters of whose ends are one and two mm. respectively. A drop of water condenses in the tube and obstructs it. Calculate the difference in pressure in the vessels compatible with the equilibrium of the drop, the angle of the cone being 15°. (Take $T = 80$ C.G.S. units.)

7. Show how the surface tension of mercury may be obtained by observation on a large drop of mercury resting on a horizontal glass plate.

8. A large drop of mercury is placed upon a horizontal solid plate. The depth of the longest diameter below the highest point of the drop is $\frac{\sqrt{3}}{2}$ of the height of the drop. Show that the angle of contact of the drop with the plate is $\cos^{-1}\frac{1}{8}$.

9. A drop of water is placed between two glass plates which are pressed together so that the drop is squeezed into a film whose area is A, and whose thickness is t. The plates will now stick together. Explain this, and, supposing one of the plates fixed, find the force which must now be applied to the other to pull it away.

10. Give an account of the formation of drops by the breaking up of a jet of liquid.

TABLES.

(1) Units and Conversion Factors.

1 inch = 2·540 centimetres	1 centimetre = ·3937 inch
1 mile = 1·609 kilometre	1 kilometre = ·6214 mile
1 cubic foot = 28·32 litres	1 litre = ·03531 cubic foot
1 gallon = 4·536 litres	= ·2200 gallon
1 pound = 453·6 grams	1 gram = ·002205 pound
1 ton = 1016 kilograms	1 tonne = ·9842 ton
1 pound-wt. = 445000 dynes	

1 radian = 57° 17′ 45″ = 206,265″
1 mean solar day = 86,400 mean solar seconds
= 86,164 sidereal seconds

1 micron (μ) = 10^{-3} millimetre = 10^{-6} metre
1 $\mu\mu$ = 10^{-6} millimetre = 10^{-9} metre
1 tenth-metre = 10^{-10} metre = $\frac{1}{10}$ $\mu\mu$
1 horse-power = 746 watts
1 atmosphere = 14·7 pds.-wt. per sq. in.
= 1·014 × 10^6 dynes per sq. cm.

(2) Density.

The litre is defined as the volume of 1 kilogram of water at 4° C. The volume of 1 gram of water at 4° C. is 1 cubic centimetre (very nearly), really 1·000028cc.

1 cubic foot of water at 62° F. weighs 62·32 pounds
1 gallon of water at 62° F. weighs 10 pounds

Solids.	gm. per c.c.
Aluminium	2·6
Brass	8·4
Cork	·24
Glass (crown)	2·5
,, (flint)	3·1
Gold	19·3
India-rubber	·95
Iron (cast)	7·2
,, (wrought)	7·8
Lead	11·4
Quartz	2·6

Liquids.	gm. per c.c.
Alcohol	·8
Glycerine	1·3
Mercury (at 0° C.)	13·596
Petroleum Oil	·8
Sulphuric Acid	1·8

Gases.	gm. per litre at 760 mm. in lat 45°
Air (at 0° C.	1·293
Hydrogen (at 0° C.)	·08987
Steam (at 100° C.)	·581

(3) Elasticity Moduli.

Substance.	Young's Modulus. Dynes per sq. cm.	Rigidity Modulus. Dynes per sq. cm.	Bulk Modulus. Dynes per sq. cm.
Brass (drawn)	1×10^{12}	$\cdot4 \times 10^{12}$	1×10^{12}
Iron (cast)	$1\cdot2 \times 10^{12}$	$\cdot4 \times 10^{12}$	1×10^{12}
Steel	2×10^{12}	$\cdot8 \times 10^{12}$	$1\cdot8 \times 10^{12}$
Glass	$\cdot7 \times 10^{12}$	$\cdot3 \times 10^{12}$	$\cdot5 \times 10^{12}$
India-rubber	50×10^{6}	—	—
Water	—	—	$\cdot02 \times 10^{12}$
Mercury	—	—	$\cdot25 \times 10^{12}$

(4) Breaking Stress.

Load in kilograms per sq. millimetre necessary to break wire of the substances named.

Lead	2	Brass	40
Silver	29	Quartz fibre	100
Copper	40	Steel (pianoforte)	200

(5) Coefficient of Restitution.

Clay	·2	Cast-iron balls	·7
Cork	·7	Glass—Brass	·8
Ivory	·8	Glass—Cast-iron	·9

(6) Coefficient of Viscosity.

Dynes per sq. cm. per unit velocity gradient.

Liquids (at 20° C.).		*Gases* (at 15° C.).	
Ether	·0023	Hydrogen	·000089
Water	·010	Air	·000181
Alcohol	·012	Carbonic acid	·000146
Mercury	·016	Carbonic oxide	·000179
Machine oil	1·0	Oxygen	·000195
Glycerine	8·5	Argon	·000221

(7) Coefficients of Diffusion from Gas to Gas.

At 0° C. and 760 mm.

Carbonic acid to air	·14 cm^2/sec.
,, ,, hydrogen	·55 ,,
Oxygen to hydrogen	·72 ,,
Hydrogen to oxygen	·68 ,,
Air to oxygen	·18 ,,
Air to hydrogen	·66 ,,

(8) Coefficient of Diffusion of Dissolved Salts.

The table gives numbers proportional to the coefficient of diffusion of aqueous solutions of salts and acids of equal molecular strengths.

Magnesium sulphate	3	Magnesium chloride	4
Sodium nitrate	5	Sodium chloride	6
Potassium bromide	8	Potassium chloride	8
Hydrochloric acid	11	Sulphuric acid	6

The diffusibility of nitric acid is nearly the same as that of hydrochloric acid, and the absolute value for a dilute solution at 90° C. is about 2×10^{-5} cm.^{2} sec.

(9) Lowering of Freezing-Point.

Per gram-molecule of salt dissolved in one litre of solvent.

Solvent.	Depression for Organic Salts and Weak Acids.	Depression for Strong Acids and Bases.
Water	1·9	3·7
Acetic Acid	3·9	—
Benzene	4·9	—

(10) Constants of Gas Particles.

(At normal pressure.)

	Velocity of Mean Square cms. per sec. at 0° C.	Mean Free Path of Molecules. cm.	Frequency of Molecular Collisions. per sec.	Diameter of Molecules. cm.
Hydrogen ...	$18·4 \times 10^4$	$18·3 \times 10^{-6}$	$10·1 \times 10^9$	$2·4 \times 10^{-8}$
Oxygen	$4·6 \times 10^4$	$10·0 \times 10^{-6}$	$4·6 \times 10^9$	$3·1 \times 10^{-8}$
Nitrogen	$4·9 \times 10^4$	$9·4 \times 10^{-6}$	$5·2 \times 10^9$	$3·3 \times 10^{-8}$
Carbonic Acid...	$3·9 \times 10^4$	$6·3 \times 10^{-6}$	$6·2 \times 10^9$	4×10^{-8}
Carbonic Oxide	$4·9 \times 10^4$	$9·3 \times 10^{-6}$	$5·2 \times 10^9$	$3·5 \times 10^{-8}$

Number of molecules in 1 c.c. of any gas at 0° C. and 760 mm.
$= 2{\cdot}75 \times 10^{19}$.
Number of atoms in 1 gm. of hydrogen $= 6{\cdot}1 \times 10^{23}$.
Mass of an atom of hydrogen $= 1{\cdot}64 \times 10^{-24}$ gm.

(11) Coefficient of Friction.

Iron on iron, ·16.
Metal on oak (fibres parallel to the motion), ·5
Oak on oak (,, ,, ,, ,,), ·5
,, ,, ,, (,, perpendicular to the motion), ·3
,, ,, ,, (,, endwise ,, ,,), ·2

Surface Tension (*in dynes per cm*).

Substance.	Δ	Temp.	In contact with Air.	In contact with Water.	In contact with Mercury.	Angle of contact with glass and air.
Water	1	15°	74	0	430	0
Mercury	13·5	17	550	430	0	52° 40′
Ethyl Alcohol ...	·80	20	22	0	400	0
Carbon Disulphide	1·27	19	34	—	—	—
Chloroform	1·49	15	27	30	400	0
Turpentine	·89	15	27	—	—	17°
Olive Oil	·91	20	32	21	—	—
Paraffin Oil	·85	25	26	48	—	26°
Solution of Sodium Carbonate	1·15	15	78	—	—	—

ANSWERS.

CHAPTER I.

5. 99·6 ft. poundals. **6.** 14·2 pounds per sq. in.

7.

	M.	L.	T.
Power	1	2	−3
Surface density ..	1	−2	0
Specific gravity ..	0	0	0
Angular velocity..	0	0	−1

9. 981 cm.

CHAPTER II.

1. $\frac{d^2}{2h}+\frac{h}{2}$. **2.** 1 %, 2 %. **6.** 1·4 in.

CHAPTER III.

3. 8000, 9000 poundals. **6.** ·066 gramme.

8. 3·2 ft. sec.$^{-2}$; 518,400 ft. pounds; 1920, 3840, 5760 ft. pounds per sec.

9. 15·9 cm. sec.$^{-1}$; $1{\cdot}63\times10^4$.

CHAPTER IV.

2. 250 c.c. **3.** 1 : 10·4, Lead; ·825 of wt. of lead.

4. 1 1·02. **5.** 2·975; 4·76 grammes.

6. Tension decreases, thrust increases by 66⅔ grams. wt. **8.** 60 c.c.

CHAPTER V.

2. 4400 ft. lb. sec.$^{-3}$; 1½ H.P. **8.** $pv \log_e 2$.

9. Efficiency × velocity ratio = mechanical advantage.

CHAPTER VI.

3. 13 ft. pounds. **4.** 13·8 pound-wts.; 7·23 sec.

5. (1) $\frac{25\pi}{3}$ radians per sec.; (2) $3{\cdot}07\times10^7$ ft. poundals; (3) 89600 lb. ft.^{2}

7. 5×10^9 ergs. **11.** $200\pi^2/(9\lambda g-200\pi^2)$.

CHAPTER VII.

2. 28·8 cm. from middle. **3.** 42 min. **5.** 100·025 (see Art. 142).
6. On the square of the amplitude if the arc is small.
11. 24·6 sec. gain.

CHAPTER VIII.

2. 1 : 1·002738.

CHAPTER IX.

2. 532 cm. sec.$^{-1}$; 50000 ergs. **4.** 5·6″. **6.** $1{\cdot}2\times10^{12}$
7. 78·5 kg. **8.** 11 cm.
9. ·004 tonnes wt.; ·002 cm. tonnes. **10.** 1·6″.
12. $9{\cdot}1\times10^3$ gram. cm.$^{-2}$. **13.** 5 in. **16.** 5 tons, 1/720.
18. $M\dfrac{\pi^2 l^2}{t_1^2-t_2^2}$, where M is the difference in mass between a solid and a hollow cylinder.

CHAPTER X.

4. $2G\rho/h$. **5.** $2\pi\rho G$.
8. Difference = ·04 tons wt. **10.** $6{\cdot}7\times10^{-8}\times m$ cm.

CHAPTER XI.

1. Take one gramme molecule measure P in gram. wt. per. sq. cm. and V in c.c.; then $R = 8{\cdot}46\times10^4$. **5.** 1·5 mm.
9. A fall of ⅜ in. **10.** $h(1 . 00016t)$. **12.** 30″. **14.** 3·2 cm.

CHAPTER XII.

1. 69·3. **2.** 100 kg. wt.; 325 kg. wt. **3.** 4·33 cm.
4. 21·4 kg. wt. **5.** 8·4 kg. wt.; 4·8 kg. wt.
6. $h_2 > 1{\cdot}6h_1$ (Fig. 106). **8.** 2·562 cub. ft.
9. 8 : 1 by weight. **10.** 12·6 kg. wt.; 4·2 kg. wt.

CHAPTER XIV.

2. g tan 20°. **3.** 210 lbs. **4.** $\dfrac{1}{\pi}\log_e 3,\ 15e^{-3\mu\pi}$. **5.** 1 in 12.

CHAPTER XV.

12. 2 : 1. **14.** 900 ergs. **16.** $2{\cdot}5\times10^5$ ergs. **18.** 1·47 cm.
19. 32 dynes per sq. cm. above atmospheric.
20. 36000 lbs. wt. per ft., 72000 lbs. wt. per ft.
21. 400 lbs. per sq. in.

ANSWERS TO THE MISCELLANEOUS EXAMPLES.

Chapter I.

1. 11,500 dynes. 2. 746.

3. $9 \cdot 18 \times 10^7$ cm., $1 \cdot 05 \times 10^{55}$ gm., $3 \cdot 06 \times 10^7$ sec.

Chapter II.

1. 140 mm. 2. 59 divisions of limb divided into 60 equal parts.

3. 39·27, 38 and 38·8 sq. cm.

Chapter III.

1. 1 : 10. 2. $7 \cdot 95 \times 10^8$ dynes.

3. 31·3 poundal-secs., 15,600 poundals.

Chapter IV.

3. $\frac{1}{9 \cdot 38}$. 4. ·0705. 5. 96·8 gm. No.

Chapter V.

2. 3830 kilogm.-wt. 4. H.P. = 28·6, 3400 lbs. 5. Pound-feet.

6. Electrical energy is 37 times as dear as coal energy.

Chapter VI.

1. $\frac{2}{3}$ in. 2. 67. 3. 0, 960, 1920 poundals.

4. 2040. 5. $\frac{7}{10}mv^2$, $996 \frac{\text{cm.}}{\text{sec.}^2}$.

Chapter VII.

1. $t = 2\pi \sqrt{\dfrac{\frac{1}{3}ml^2 + \frac{2}{5}Mr^2 + M(l+r)^2}{mg\frac{l}{2} + Mg(l+r)}}$. 4. $t = 2\pi \sqrt{\dfrac{l}{3g}}$.

5. $t = 2\pi \sqrt{\dfrac{a}{g}}$. 6. Radius $= \frac{5}{28} \dfrac{gT^2}{\pi^2} + r$.

8. $\dfrac{T_1 T_2}{\sqrt{T_1^2 + T_2^2}}$.

Chapter VIII.

3. $-10'$, morning $20'$ longer.

Chapter IX.

2. $1 \cdot 02 \times 10^6$ cm. 4. $\frac{1}{3}$. 5. From 1 to 5 %.
8. $9 \cdot 6 \times 10^{11}$ dynes/cm.^{2}. 10. ·055″. 11. 48 dyne-cms.
12. 5·27 cm. 13. 70 cms. per sec. 14. ·7.

Chapter X.

1. 988 cms./sec.^{2}. 4. Lose 144. 5. 17 times.
6. ·5051 sec.

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3. ·0122 dyne per sq. cm. per unit velocity gradient.
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1. 11·2 pds.-wt., 385. 2. $9 \cdot 7 \times 10^{10}$ ergs.
3. $\frac{12}{49}\frac{mu^2}{F}$. 4. 3·55 gm., 1014 gm.

Chapter XV.

3. 69 dynes/cm.
4. $h = \frac{l}{2} \pm \sqrt{\frac{l^2}{4} - \frac{2Tl}{rdg}}$. T must be $> \frac{rdgl}{2}$. 5. ·6 cm.
6. 1590 dynes/cm.^{2}. 9. $\frac{2AT}{t}$ dynes.

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